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Making holes in the hyperspace of subcontinua of smooth dendroids

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Abstract

A *continuum* is a non-degenerate compact connected metric space. Let $C(X)$ be the hyperspace of all subcontinua of X. An element $A \in C(X)$ *makes a hole* in $C(X)$ if $C(X) - \{A\}$ is not unicoherent. In this paper, we characterize the elements $A \in C(X)$ satisfying that A makes a hole in $C(X)$ when X is a smooth dendroid.

Keywords: Continuum, Hyperspace of subcontinua, Property b), Smooth dendroid, Unicoherence, Whitney levels

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1. Introduction

A connected topological space Z is *unicoherent* if whenever $Z = A \cup B$, where A and B are connected closed subsets of Z, then the set $A \cap B$ is connected. An element z of a unicoherent topological space Z *makes a hole in* Z if $Z - \{z\}$ is not unicoherent.

A *continuum* is a non-degenerate compact connected metric space. Given a continuum X, the hyperspace of all nonempty subcontinua of X is denoted by $C(X)$ metrized by the Hausdorff metric (see, [11, Definition 2.1, p. 11]). In [11,

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Theorem 19.8, p. 159] Sam B. Nadler, Jr. proves that the hyperspace $C(X)$ is unicoherent for any continuum X.

In this paper we are interested in the following problem which arises in [1, p. 2000]:

Problem. Let $\mathcal{H}(X)$ be a hyperspace of X. For which elements $A \in \mathcal{H}(X)$, A makes a hole in $\mathcal{H}(X)$.

The classification of the points that make a hole in a unicoherent space has been useful to distinguish topological spaces, specially hyperspaces, for example: in [10, Lemmas 2.1 - 2.2, p. 348-349] A. Illanes shows that $C_2([0,1]) - \{A\}$ is unicoherent for each $A \in C_2([0,1])$ (where $C_2(X)$ is the hyperspace of all non-empty closed subsets of a continuum X having at most two components) while $C_2(S) - \{S\}$ is not unicoherent, where S is a simple closed curve. As a consequence the author obtains that $C_2([0,1])$ and $C_2(S)$ are not homeomorphic; this in contrast to the fact that $C([0,1])$ and $C(S)$ are homeomorphic.

In the current paper, we present the solution to the problem when X is a smooth dendroid and $\mathcal{H}(X) = C(X)$. Our main result generalizes to [3, Theorem 3.8, p. 136].

Readers specially interested in this problem are referred to [1]-[6].

2. Auxiliary results

We use the symbols $\mathbb{N}$ and $\mathbb{R}$ to denote the set of all positive integers and the set of all real numbers, respectively.

For a subset W of a topological space Z, $Comp(W)$ will represent the set of all component of W. A point z in a connected topological space Z is a *cut point* of Z provided that $Z - \{z\}$ is not connected.

An *arc* is any homeomorphic space to the closed unit interval [0,1].

The word *map* stands for a continuous function between topological spaces.

A subspace Y of a topological space Z is a *deformation retract* of Z if there exists a map $H : Z \times [0,1] \to Z$ such that $H(z,0) = z$ for each $z \in Z$,

$H(Z \times \{1\}) = Y$ and $H(y,1) = y$ for each $y \in Y$. A topological space Z is *contractible* if exists $z \in Z$ in such a way that $\{z\}$ is a deformation retract of Z.

A map f from a connected topological space Z into the unit circumference centered at the origin in the Euclidean plane S^1 *has a lifting* if there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$, where $\exp : \mathbb{R} \to S^1$ is defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A connected topological space Z *has property* $b)$ if each map from Z into S^1 has a lifting. Observe that to have property b) is topological property.

It is known that each metric space having property b) is unicoherent (see [12, Theorem 7.3, p. 227]). This fact will be used repeatedly without mentioning explicitly throughout this paper. Consequently, to have property b) will be an important tool to obtain the desired classification. For this, we present known results in the literature which we will use frequently:

Proposition 2.1. *[1, Proposition 8, p. 2001] Let Z be a topological space and let W and Y be non-empty closed subsets of Z such that $Z = W \cup Y$. If W and Y have property b) and $W \cap Y$ is connected, then Z has property b).*

Proposition 2.2. *[1, Proposition 9, p. 2001] Let Z be a topological space connected and let Y be a deformation retract of Z. Then Z has property (b) if and only if Y has property (b).*

An immediately consequence of previous proposition is the following result.

Corollary 2.3. *Each contractible metric space has property b). In particular, each contractible metric space is unicoherent.*

Given a continuum X, $F_1(X)$ denotes the hyperspace of all degenerate subcontinua of X, this is $F_1(X) = \{\{x\} : x \in X\}$.

A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0,1]$ such that:

1. $\mu(\{x\}) = 0$ for each $x \in X$,

2. $\mu(A) < \mu(B)$ if $A \subset B$ and $A \neq B$,

3. $\mu(X) = 1$.

It is known that Whitney maps always exist (see [11, Theorem 13.4, p. 107]). A *Whitney level* is a subspace of $C(X)$ of the form $\mu^{-1}(t)$ where $0 < t < 1$ and μ is a Whitney map for $C(X)$.

The result below follows from [1, Lemma 13, p. 2004].

Proposition 2.4. *Let X be a continuum, let μ be a Whitney map for $C(X)$ and let $A \in C(X) - \{X\}$. Then $\mu^{-1}([\mu(A), 1]) - \{A\}$ has property b).*

The next characterization of the cut points of Whitney levels will be used frequently in the proof of our main theorems.

Proposition 2.5. *[9, Theorem 2.1, p. 210] Let X be a continuum, let $A \in C(X)$, let μ be a Whitney map for $C(X)$ and let $t = \mu(A)$. Then, A is a cut point of $\mu^{-1}(t)$ if and only if there exist non-empty disjoint open subsets U and V of X such that $X - A = U \cup V$ and each $B \in \mu^{-1}(t)$ satisfies either $B \subset U \cup A$ or $B \subset V \cup A$.*

For a continuum X, an *order arc* in C(X) is an arc α in $C(X)$ such that if $A, B \in \alpha$, then either $A \subset B$ or $B \subset A$. If α is an order arc in $C(X)$, then α is said to be *an order arc from $\bigcap \alpha$ to $\bigcup \alpha$.*

Each non-degenerate proper subcontinuum of a continuum will be called *non-trivial*.

3. Smooth dendroids

A *dendroid* is a hereditarily unicoherent arcwise connected continuum (hereditarily unicoherent means each one of its subcontinua is unicoherent). Each subcontinuum of a dendroid is a dendroid. Let X be a dendroid. A point $x \in X$ will be called *end point* of X provided that x is not a cut point of any arc in X containing it. The set of all end points of X is denoted by $E(X)$. Each point $x \in X$ which is a common end point of at least three different arcs is called *ramification point* of X. The symbol $R(X)$ represents the set of all ramification points of X. If $x, y \in X$ are such that $x \neq y$, then $[x, y]$ will denote the unique arc in X whose end points are x and y. Set $[x, x] = \{x\}$ for each $x \in X$.

A dendroid X is said to be *smooth* at $p \in X$ provided that for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to a point $x \in X$, the sequence of arcs $\{[p, x_n]\}_{n\in\mathbb{N}}$ converges to $[p, x]$ in $C(X)$.

Throughout this paper X will denote a smooth dendroid at p and μ will denote a Whitney map for $C(X)$.

Let B a subset of X, let $C \in Comp(X - B)$ and let $b \in B$. We say that b is *arcwise accessible from* C provided that there exists an arc in $C \cup \{b\}$ having b as an end point.

Lemma 3.1. *Let $A \in C(X)$ be non-trivial. If C_1, $C_2 \in Comp(X - A)$ and there exist $a, b \in A$ such that $[a, b]$ a proper subcontinuum of A and a, b are arcwise accessible from C_1 and C_2, respectively, then there exists $J \in \mu^{-1}(\mu(A))$ satisfying that $C_1 \cap J \neq \emptyset$ and $C_2 \cap J \neq \emptyset$.*

Proof. Let $t = \mu(A)$. Then $\mu([a, b]) < t$. Since a is arcwise accessible from C_1, there exists an arc W such that $W \subset C_1 \cup \{a\}$ and a is an end point of W. Set $F = [a, b] \cup W$. Notice that $F \in C(X)$. By [11, Theorem 14.6, p. 112], there exists an order arc α in $C(X)$ from $[a, b]$ to F. Fix $s \in [0, 1]$ in such a way $\mu([a, b]) < s < \min\{t, \mu(F)\}$. Then, there exists $G \in \alpha$ satisfying that $\mu(G) = s$. Observe that $G \cap C_1 \neq \emptyset$. Now, from our assumption there exists an arc Y such that $Y \subset C_2 \cup \{b\}$ and b is an end point of Y. Set $I = G \cup Y$. We have that $I \in C(X)$. Thus, there exists an order arc β in $C(X)$ from G to X fulfilling that $I \in \beta$. We may take $J \in \beta \cap \mu^{-1}(t)$. Note that $C_2 \cap J \neq \emptyset$ and that the inclusion $G \subset J$ guarantees that $C_1 \cap J \neq \emptyset$. Thus, J satisfies all our requirements.

Define *the partial order* $\leq_p$ by letting $x \leq_p y$ whenever $[p, x] \subset [p, y]$. Let $\rho : C(X) \to X$ be defined by $\rho(B)$ is the unique zero of B relative to $\leq_p$. The function ρ is continuous (see [8, Theorem I5-A, p. 552]) and it satisfies that $\rho(B) \in B$ and B is a smooth dendroid at $\rho(B)$ for each $B \in C(X)$.

Given $B \in C(X)$, define $g_B : B \times [0,1] \to B$ by $g_B(x,t)$ is the unique point of $[\rho(B), x]$ such that $\mu([\rho(B), g_B(x,t)]) = (1-t)\mu([\rho(B), x])$.

Proposition 3.2. *For each $B \in C(X)$, each one of the following conditions holds.*

1. *g_B is well defined,*
2. *g_B is a continuous,*
3. *for each $x \in B$, $g_B(x,0) = x$ and $g_B(x,1) = \rho(B)$*

Proof. Let $B \in C(X)$ be arbitrary. Set $b = \rho(B)$.

(1) Let $(x,t) \in B \times [0,1]$ be arbitrary. Define $\mathcal{A} = \{[b,z] : z \in [b,x]\}$. Note that $\mathcal{A}$ is an arc in $C(B)$ whose end points are $\{b\}$ and $[b,x]$. Since $0 = \mu(\{b\}) \leq (1-t)\mu([b,x]) \leq \mu([b,x])$, by the continuity of the one-to-one function $\mu|_{\mathcal{A}}$, there exists a unique point $g_B(x,t) \in [b,x]$ such that $\mu([b, g_B(x,t)]) = (1-t)\mu([b,x])$. Therefore, g_B is well defined.

(2) In order to prove the continuity of g_B, let $\{(x_n, t_n)\}_{n\in\mathbb{N}}$ be a sequence converging to (x,t) in $B \times [0,1]$. We may suppose that there exists $y \in B$ satisfying that $y = \lim g_B(x_n, t_n)$. Next, we will show $y = g_B(x,t)$. Since $y = \lim g_B(x_n, t_n)$, each $g_B(x_n, t_n) \in [b, x_n]$ and B is a smooth dendroid at b, we deduce that $y \in \lim[b, x_n] = [b,x]$ and $[b,y] = \lim[b, g_B(x_n, t_n)]$. So, from the continuity of μ, it follows that $\mu([b,y]) = \lim \mu([b, g_B(x_n,t_n)]) = \lim(1 - t_n)\mu([b,x_n]) = (1-t)\mu([b,x])$. Thus $y = g_B(x,t)$.

(3) Let $x \in B$ be arbitrary. The definition of g_B guarantees $\mu([b, g_B(x,0)]) = (1-0)\mu([b,x]) = \mu([b,x])$. This and the inclusion $[b, g(x,0)] \subset [b,x]$ guarantee that $[b, g_B(x,0)] = [b,x]$. Hence, $g_B(x,0) = x$. Now, $g_B(x,1)$ is the unique point of $[b,x]$ such that $\mu([b, g_B(x,1)]) = (1-1)\mu([b,x]) = 0$. This implies that $b = g_B(x,1)$.

The map g_B will be used constantly in this paper without mentioning its definition explicitly.

Corollary 3.3. *The smooth dendroid at p X has property b).*

Proof. The map g_X satisfies that $g_X(x, 0) = x$ and $g_X(x, 1) = p$ for each $x \in X$ (see, (3) of Proposition 3.2). Thus, X is contractible and, by Proposition 2.3, X has property b).

Lemma 3.4. *Let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence converging to B in $C(X)$ and let $\{(y_n, l_n)\}_{n\in\mathbb{N}}$ be a sequence converging to (y, l) in $X \times [0, 1]$. If $y_n \in B_n$ for each $n \in \mathbb{N}$, then the sequence $\{g_{B_n}(y_n, l_n)\}_{n\in\mathbb{N}}$ converges to $g_B(y, l)$.*

Proof. For each $n \in \mathbb{N}$, set $x_n = g_{B_n}(y_n, l_n)$. We may assume that there exists $x \in X$ such that $x = \lim x_n$. Since $x_n \in [p, y_n]$ for every $n \in \mathbb{N}$, by our assumption X is a smooth dendroid at p, we have that $x \in [p, y]$. Now, set $b = \rho(B)$ and $b_n = \rho(B_n)$ for each $n \in \mathbb{N}$. From the definition of ρ, it follows that each $b_n \in [p, x_n]$ and each $b_n \in [p, y_n]$. Thus, the continuity of ρ and [7, Theorem 12, p. 312] guarantee that

$$[b, x] = \lim[b_n, x_n] \text{ and } [b, y] = \lim[b_n, y_n]. \tag{1}$$

On the other hand, by definition of g_{B_n}, for each $n \in \mathrm{N}$, we have that $x_n \in [b_n, y_n]$ and $\mu([b_n, x_n]) = (1 - l_n)\mu([b_n, y_n])$. By the continuity of μ and (1), we obtain that $\mu([b, x]) = (1 - l)\mu([b, y])$. This and the fact that $g_B(y, l)$ is the unique point of $[p, y]$ such that $\mu([b, g_B(y, l)]) = (1 - l)\mu([b, y])$ imply $g_B(y, l) = x$.

Proposition 3.5. *Let $A \in C(X) - F_1(X)$. Then $F_1(X)$ is a deformation retract of $\mu^{-1}([0, \mu(A)]) - \{A\}$.*

Proof. Set $W = \mu^{-1}([0, \mu(A)]) - \{A\}$. Define $H : W \times [0, 1] \to W$ by

$$H(B, l) = g_B(B \times \{l\})$$

First, we are going to prove that H is well defined. Let $(B, l) \in W \times [0, 1]$ be arbitrary. From (2) of Proposition 3.2, we deduce that $H(B, l) \in C(X)$. Now, observe that $H(B, l) \subset B$. Then $\mu(H(B, l)) \leq \mu(B) \leq \mu(A)$ and so

$H(B,l) \in \mu^{-1}([0,\mu(A)])$. Next, assume that $H(B,l) = A$. From this, we obtain that $A \subset B$ and $\mu(A) \leq \mu(B)$. Thus, $\mu(A) = \mu(B)$ and so $A = B$, a contradiction. This proves that H is well defined.

In order to show that H is continuous, let $\{(B_n, l_n)\}_{n\in\mathbb{N}}$ be a sequence converging to (B,l) in $W \times [0,1]$. We may assume that there exists $F \in W$ such that $F = \lim H(B_n, l_n)$. Let us prove that $H(B,l) = F$.

Let $x \in H(B,l)$ be arbitrary. Then there exists $y \in B$ such that $x = g_B(y,l)$. Since $B = \lim B_n$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ converging to y in X such that $y_n \in B_n$ for each $n \in \mathbb{N}$. Invoke Lemma 3.4 to prove that $\lim g_{B_n}(y_n, l_n) = g_B(y,l) = x$, and so $x \in F$.

Now, let $z \in F$. Then there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ in X such that $w_n \in B_n$ for each $n \in \mathbb{N}$ and $z = \lim g_{B_n}(w_n, l_n)$. Taking subsequence if it is necessary, we may assume that exists $w \in X$ such that $w = \lim w_n$. So, $w \in B$. Using Lemma 3.4, we obtain that $z = \lim g_{B_n}(w_n, l_n) = g_B(w,l)$. This shows that $z \in H(B,l)$. In conclusion the continuity of H holds.

Finally, from (3) of Proposition 3.2, we have $H(B,0) = g_B(B\times\{0\}) = B$ and $H(B,1) = g_B(B \times \{1\}) = \{\rho(B)\} \in F_1(X)$ for each $B \in W$, and $H(\{x\},1) = \{x\}$ for all $x \in X$. Therefore, $F_1(X)$ is a deformation retract of W.

Theorem 3.6. *Let $A \in C(X)-F_1(X)$. Then, $\mu^{-1}([0,\mu(A)])-\{A\}$ has property b).*

Proof. It is known that X is homeomorphic to $F_1(X)$. Then $F_1(X)$ has property b) (see Corollary 3.3). Now, from Proposition 2.2 and Proposition 3.5, we deduce that $\mu^{-1}([0,\mu(A)]) - \{A\}$ has property b).

4. Main Theorems

Throughout this section A denotes an element of $C(X)$.

Theorem 4.1. *If $A \notin F_1(X)$ and $A \cap E(X) - \{p\} \neq \emptyset$, then A does not make a hole in $C(X)$.*

PROOF. In order to prove that $C(X) - \{A\}$ is unicoherent, by Corollary 2.3, it is suffices to show that $C(X) - \{A\}$ is contractible.

Define $H : (C(X) - \{A\}) \times [0, 1] \to C(X) - \{A\}$ by

$$H(B, t) = g_X(B \times \{t\})$$

From (3) of Proposition 3.2, it follows that

$$H(B, 0) = g_X(B \times \{0\}) = B \text{ and } H(B, 1) = g_X(B \times \{1\}) = \{p\} \tag{2}$$

for each $B \in C(X) - \{A\}$. In order to prove that H is a map, let us start by proving that H is well defined. To this end, let $(B, t) \in (C(X) - \{A\}) \times [0, 1]$ be arbitrary. First, the continuity of g_X and the inclusion $B \in C(X)$ guarantee that $g_X(B\times\{t\}) = H(B, t) \in C(X)$. Second, suppose that $H(B, t) = A$. By (2) we obtain that $0 < t < 1$. Now, let $e \in A\cap E(X)-\{p\}$. Then there exists $x \in B$ such that $g_X(x, t) = e$. Thus, by definition of g_X, we deduce that $e \in [p, x]$ and $\mu([p, e]) = (1 - t)\mu([p, x]) < \mu([p, x])$. This implies that $[p, e]$ is a proper subset of $[p, x]$, a contradiction. In conclusion, $H(B, t) \in C(X) - \{A\}$.

Finally, the continuity of g_X guarantees that of H. Hence, $C(X) - \{A\}$ is contractible.

An immediately consequence of our previous theorem is the next result.

Corollary 4.2. *The element X of $C(X)$ does not make a hole in $C(X)$.*

The theorem below presents a characterization of non-trivial subcontinua that make a hole in $C(X)$ in terms of Whitney level containing it. This characterization will aid to prove our main results.

Theorem 4.3. *Let $t = \mu(A)$. If A is non-trivial, then A does not make a hole in $C(X)$ if and only if $\mu^{-1}(t) - \{A\}$ is connected.*

PROOF. First, set $W = \mu^{-1}([0, t]) - \{A\}$ and $Y = \mu^{-1}([t, 1]) - \{A\}$. Observe that W and Y are connected closed subsets of $C(X)-\{A\}$, $C(X)-\{A\} = W\cup Y$, $W \cap Y = \mu^{-1}(t) - \{A\}$ and, by Proposition 2.4 and Theorem 3.6, W and Y have property b).

Now, if we assume that A does not make a hole in $C(X)$, then $C(X) - \{A\}$ is unicoherent and so, $W \cap Y = \mu^{-1}(t) - \{A\}$ must be connected.

Finally, when $W \cap Y = \mu^{-1}(t) - \{A\}$ is connected, by Proposition 2.1, $C(X) - \{A\}$ has property b). Hence, $C(X) - \{A\}$ is unicoherent.

Theorem 4.4. *If A is non-trivial and $X - A$ is connected, then A does not make a hole in $C(X)$.*

Proof. From our assumption and Propositions 2.5, it follows that $\mu^{-1}(\mu(A)) - \{A\}$ is connected. Thus, by Theorem 4.3, A does not make a hole in $C(X)$.

Theorem 4.5. *If A is non-trivial, A is not an arc and $X - A$ is not connected, then A does not make a hole in $C(X)$.*

Proof. Let $t = \mu(A)$. In light of Theorem 4.3, it suffices to prove that $\mu^{-1}(t) - \{A\}$ is connected. Suppose to the contrary that $\mu^{-1}(t) - \{A\}$ is not connected. By Proposition 2.5, there exist disjoint non-empty open subsets U and V of X such that $X - A = U \cup V$ and each $B \in \mu^{-1}(t)$ satisfies either $B \subset U \cup A$ or $B \subset V \cup A$.

Now, let C_1, $C_2 \in Comp(X - A)$ be such that $C_1 \subset U$ and $C_2 \subset V$. Taking $r \in C_1$ and $q \in C_2$, we have that $[r, q] \cap A \neq \emptyset$. Next, let $h : [0, 1] \to [r, q]$ be a homeomorphism such that $h(0) = r$ and $h(1) = q$. Define $t_0 = \inf\{t \in [0, 1] : h(t) \in A\}$ and set $a = h(t_0)$. Thus, $h([0, t_0]) = [h(0), h(t_0)] = [r, a] \subset C_1 \cup \{a\}$. Then a is arcwise accessible from C_1. Similarly, define $t_1 = \sup\{t \in [0, 1] : h(t) \in A\}$ and $b = h(t_1)$ to get that b is arcwise accessible from C_2. Since $[a, b]$ must be a proper subcontinuum of A, Lemma 3.1 guarantees the existence of $J \in \mu^{-1}(t)$ satisfying that $J \cap C_1 \neq \emptyset$ and $J \cap C_2 \neq \emptyset$; hence $J \cap U \neq \emptyset$ and $J \cap V \neq \emptyset$. This contradicts the choice of U and V. Therefore, $\mu^{-1}(t) - \{A\}$ is connected and so, A does not make a hole in $C(X)$.

Theorem 4.6. *If A is a non-trivial arc, $X - A$ is not connected and $p \in A \cap E(X)$, then A does not make a hole in $C(X)$.*

Proof. Set $t = \mu(A)$. We are going to prove that $\mu^{-1}(t) - \{A\}$ is connected. Seeking a contradiction assume that $A = [p, b]$ and A is a cut point of $\mu^{-1}(t)$. So, in light of Proposition 2.5 there exist disjoint non-empty open subset U and V of X such that $X - A = U \cup V$ and if $B \in \mu^{-1}(t)$, then either $B \subset U \cup A$ or $B \subset A \cup V$.

Fix $q \in U$ and $r \in V$. Let $C_1, C_2 \in Comp(X - A)$ be such that $q \in C_1$ and $r \in C_2$. Observe that $C_1 \subset U$, $C_2 \subset V$ and $[q, r] \cap A \neq \emptyset$. Let $h : [0, 1] \to [q, r]$ be a homeomorphism such that $h(0) = q$ and $h(1) = r$. Consider $t_0 = \inf\{t \in [0, 1] : h(t) \in A\}$, $t_1 = \sup\{t \in [0, 1] : h(t) \in A\}$ and let $a = h(t_0)$ and $c = h(t_1)$. To see that a is arcwise accessible from C_1 and c is arcwise accessible from C_2, simply note that $h([0, t_0]) = [h(0), h(t_0)] = [q, a] \subset C_1 \cup \{a\}$ and $h([t_1, 1]) = [h(t_1), h(1)] = [c, r] \subset C_2 \cup \{c\}$. From the fact that $p \in A \cap E(X)$, it follows that $p \notin [q, r]$ and so, $[a, c]$ is a proper subcontinuum of A. By Lemma 3.1, there exists $J \in \mu^{-1}(t)$ satisfying that $J \cap C_1 \neq \emptyset$ and $J \cap C_2 \neq \emptyset$, and so $J \cap U \neq \emptyset$ and $J \cap V \neq \emptyset$. This is a contradiction. Therefore, A does not make a hole in $C(X)$.

Theorem 4.7. *If A is a non-trivial arc, $X - A$ is not connected and $R(X) \cap A - E(A) \neq \emptyset$, then A does not make a hole in $C(X)$.*

Proof. Let $t = \mu(A)$. We are going to prove that $\mu^{-1}(t) - \{A\}$ is connected. Assume that not, then there exist disjoint non-empty open subsets U and V of X such that $X - A = U \cup V$ and each $B \in \mu^{-1}(t)$ satisfies either $B \subset U \cup A$ or $B \subset V \cup A$ (see Proposition 2.5). Now, we shall show that there exists $J \in \mu^{-1}(t)$ such that $J \cap U \neq \emptyset$ and $J \cap V \neq \emptyset$, this will contradict the choice of U and V.

To this end, suppose that $A = [a, b]$. Let $x \in R(X) \cap A - E(A)$. Since A is an arc, there exists an arc C in X such that x is an end point of C and $A \cap C = \{x\}$. This, if $C_1 \in Comp(X - A)$ is such that $C - \{x\} \subset C_1$, then x is arcwise accessible from C_1 and either $C_1 \subset U$ or $C_1 \subset V$. Suppose that $C_1 \subset U$.

Now, fix $w \in V$. Let $C_2 \in Comp(X - A)$ be such that $w \in C_2$. Consider a homeomorphism $h : [0,1] \to [x,w]$ satisfying that $h(0) = x$ and $h(1) = w$. Define $t_1 = \sup\{t \in [0,1] : h(t) \in A\}$ and $z = h(t_1)$. Note that $h([t_1,1]) = [h(t_1,1)] = [z,w] \subset C_2 \cup \{z\}$ and so, z is arcwise accessible de C_2. Observe that $[x,z]$ is a proper subcontinuum of A. By Lemma 3.1, there exists $J \in \mu^{-1}(t)$ satisfying that $J \cap C_1 \neq \emptyset$ and $J \cap C_2 \neq \emptyset$, and so $J \cap U \neq \emptyset$ and $J \cap V \neq \emptyset$. In conclusion, $\mu^{-1}(t) - \{A\}$ is connected and, by Theorem 4.3, A does not make a hole in $C(X)$.

For a non-trivial arc B, let $L_B = \{y \in X - B : B \subset [p,y]\}$.

Theorem 4.8. *If A is a non-trivial arc, $X - A$ is not connected, $R(X) \cap A - E(A) = \emptyset$ and L_A is not open, then A does not make a hole in $C(X)$.*

PROOF. Let $t = \mu(A)$. By Theorem 4.3, it suffices to show that $\mu^{-1}(t) - \{A\}$ is connected. If $\mu^{-1}(t) - \{A\}$ is not connected, then there exist disjoint non-empty open subsets U and V of X such that $X - A = U \cup V$, and each $B \in \mu^{-1}(t)$ satisfies either $B \subset U \cup A$ or $B \subset V \cup A$ (see Proposition 2.5). Suppose that $A = [a,b]$ and $a \in [p,b]$. Now, we shall show that there exists $J \in \mu^{-1}(t)$ fulfilling that $J \cap U \neq \emptyset$ and $J \cap V \neq \emptyset$. To this end, we considerer the following two cases.

Case 1. Either $L_A \subset U$ or $L_A \subset V$.

Assume that $L_A \subset U$. From the fact that L_A is not open, it follows that there exists $x \in U - L_A$. Take $y \in V$. So, $y \in X - L_A$. Then, from our assumption $R(X) \cap A - E(A) = \emptyset$, it follows that $[x,y] \cap A = \{a\}$. Thus, if $C_1, C_2 \in Comp(X - A)$ such that $x \in C_1 \subset U$ and $y \in C_2 \subset V$, we have that a is arcwise accessible from C_1 and C_2. Lemma 3.1 guarantees the existence of $J \in \mu^{-1}(t)$ satisfying that $J \cap C_1 \neq \emptyset$ and $J \cap C_2 \neq \emptyset$; hence $J \cap U \neq \emptyset$ and $J \cap V \neq \emptyset$.

Case 2. $L_A \cap U \neq \emptyset$ and $L_A \cap V \neq \emptyset$.

Let $x \in L_A \cap U$ and $y \in L_A \cap V$. This and the equality $R(X) \cap A - E(A) = \emptyset$ imply that $[x,y] \cap A = \{b\}$. Then, b is arcwise from C_1 and C_2, where $C_1, C_2 \in$

$Comp(X-A)$ such that $x \in C_1 \subset U$ and $x \in C_2 \subset V$. So, by Lemma 3.1, there exists $J \in \mu^{-1}(t)$ satisfying that $J \cap C_1 \neq \emptyset$ and $J \cap C_2 \neq \emptyset$. We deduce that $J \cap U \neq \emptyset$ and $J \cap V \neq \emptyset$.

In both cases, we obtain a contradiction to the choice of U and V. Therefore, $\mu^{-1}(t) - \{A\}$ is connected and so, A does not make a hole in $C(X)$.

A non-trivial arc B is called *simple arc* if $B \cap E(X) = \emptyset$, $X - B$ is not connected, $R(X) \cap B - E(B) = \emptyset$ and L_B is open. Our definition of simple arc for a smooth fan is equivalent to the definition of a simple arc given in [3, p. 134].

Theorem 4.9. *If A is a simple arc, then A makes a hole in $C(X)$.*

Proof. In light of Theorem 4.3, it suffices to prove that $\mu^{-1}(t) - \{A\}$ is not connected where $t = \mu(A)$. Set $U = L_A$ and $V = X - (A \cup L_A)$. Note that $A \cup L_A$ is closed. Then V is open. These disjoint non-empty open subsets of X satisfy that $X - A = U \cup V$. Now, we are going to prove that if $B \in \mu^{-1}(t) - \{A\}$, then either $B \subset A \cup U$ or $B \subset A \cup V$.

To this end, we suppose to the contrary that there exists $B \in \mu^{-1}(t) - \{A\}$ such that $B \cap U \neq \emptyset$ and $B \cap V \neq \emptyset$. Fix $x \in B \cap U$ and $y \in B \cap V$. Since B is arcwise connected, we have that $[x, y] \subset B$. We also have $A \subset [p, x]$ and $A \not\subset [p, y]$. Next, if $[x, y] \cap A - E(A) \neq \emptyset$, then $z \in [x, y] \cap A - E(A)$ is such that $[z, y] \cap [z, p] = \{z\}$, $[z, y] \cap [x, z] = \{z\}$ and $[p, z] \cap [x, z] = \{z\}$, and so, $z \in R(X) \cap A - E(A)$, this is a contradiction. Then $E(A) \subset [x, y]$. This imply that A is a proper subset of $[x, y]$. Thus $t = \mu(A) < \mu([x, y]) \leq \mu(B)$, a contradiction. Hence, either $B \subset A \cup U$ or $B \subset A \cup V$, and so by Proposition 2.5, $\mu^{-1}(t) - \{A\}$ is not connected. The proof is complete.

Classification

Theorem 4.10. *The subcontinuum A makes a hole in $C(X)$ if and only if A is a simple arc.*

Proof. First, assume that A makes a hole in $C(X)$. A consequence of [1, Theorem 3, p. 2001] and Corollary 4.2 is the fact that A is non-trivial. Next, Theorem 4.1 implies that $A \cap E(X) - \{p\} = \emptyset$. By Theorem 4.6, we have that $p \notin A \cap E(X)$ and hence $A \cap E(X) = \emptyset$. Now, Theorem 4.4 guarantees that $X - A$ is not connected. From Theorem 4.5, it follows that A is an arc. Using Theorem 4.7, we deduce that $R(X) \cap A - E(A) = \emptyset$. Invoke Theorem 4.8 to prove that L_A is open. In conclusion, A is a simple arc.

The converse follows from Theorem 4.9.

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