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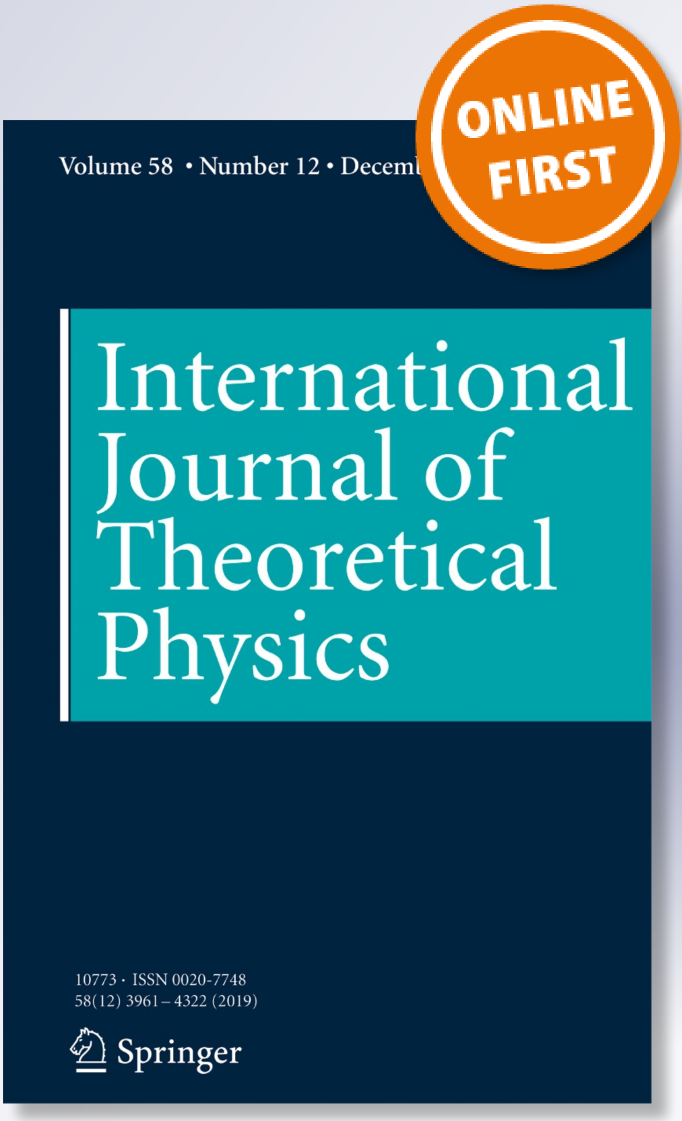
International Journal of Theoretical Physics

ISSN 0020-7748

Int J Theor Phys
DOI 10.1007/s10773-019-04294-2



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https://doi.org/10.1007/s10773-019-04294-2

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M. Ávila[1] · J. Rueda-Paz[1]

Received: 7 February 2019 / Accepted: 17 September 2019 / [redacted]

Abstract

We consider a qubit-qutrit system interacting each other through an isotropic Heisenberg Hamiltonian in a uniform magnetic field. An expression for the entanglement between the two parties as a function of both the anisotropy factor and the expansion coefficients is found. It is also calculated the uncertainty in the concurrence $\Delta\mathcal{C}$ (measured as the standard deviation) as a function of the anisotropy factor. We make a strictly mathematical assumption that there exists a canonical conjugate variable (called π) to the concurrence $\mathcal{C}$. Furthermore, we assume that such two variables satisfy (Robertson, Phys. Rev. **34**, 163 1929) uncertainty relation of the form $(\Delta A)^2(\Delta B)^2 > |\frac{1}{2}\langle\psi|[A, B]|\psi\rangle|^2$. From such an inequality we impose bounds for the uncertainty of π for several values of the anisotropy factor. The above may help to open a novel vision about the properties of the concurrence between a qubit and a qutrit.

Keywords Entanglement · Uncertainty · Qubit · Qutrit

1 Introduction

Entanglement is an ingredient which distinguishes quantum mechanics from classical mechanics. In the past years, entanglement has attracted too much the attention because it is fundamental for the processing of quantum algorithms such as those of Shor's and Grover's [2]. Entanglement differs from classical correlations in some key aspects as it is the violation of Bell's inequality [3].

The spin model is important in several branches of physics as condensed matter and NMR models too. For instance, it has been studied the entanglement of a spin system ground state and a thermal state of spin-1/2 pair with Heisenberg-type coupling in presence of a magnetic field [4, 5] where interesting results were obtained. On the other hand, the relation between quantum phase transition (QPT) and entanglement has been subjected to exhaustive studies [6]. It is well-known that near the quantum critical points in QPT, the systems can be

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simply characterized in terms of their quantity of entanglement [7]. Of particular relevance is the entanglement of the ground state of a system which may be degenerate. In the present paper, we calculate the concurrence C between a qubit (a two-level system) and a qutrit (a three-level system) coupled each other through a Heisenberg anisotropic Hamiltonian in presence of a uniform magnetic field. The concurrence between the qubit and the qutrit is derived as a function of the anisotropy of the coupling. Then we calculate the uncertainty in the concurrence ΔC (as the standard deviation) as a function of both the anisotropy factor and the coefficient of the basis vectors of the Hilbert space. Due that it is a physical quantity, the concurrence can be represented by a Hermitian operator C. In the present work we make a mathematical assumption that there exists a canonical conjugate variable to the concurrence which we call π which satisfy a Robertson's [1] uncertainty relation of the form $(\Delta\mathcal{C})^2(\Delta\pi)^2 > |\frac{1}{2}\langle\psi|[\mathcal{C},\pi]|\psi\rangle|^2$. Through the use of the above relation and calculating the uncertainty in $\mathcal{C}$ as a function of the anisotropy factor, we impose bound on the quantity $\Delta\pi$ which represents the uncertainty of the hypothetical canonical conjugate variable to the concurrence. The present assumption may help open a new vision of the concurrence though of this as a dynamical variable. The implications of the above could be important for Quantum Information Science.

In order to investigate the properties of the set of density matrices of finite size, in Refs. [8–11] it was introduced the concept of average entanglement. In [12] it was obtained an analytical expression for the probability density distribution of linear entropy and the purity for bipartite pure random quantum states. Such an expression was employed in Ref. [6] for measuring the mean value of the entanglements of degenerate ground states. All of these results are applied in the present work.

In Section 2, it is reviewed the concept of entanglement in a Hilbert subspace while in Section 3 we calculate the uncertainty in the concurrence as a function of the anisotropy of the coupling. In Section 4 we give a discussion on an uncertainty relation for the concurrence. Finally, in Conclusions, it is given a brief discussion of our results.

2 Entanglement in Hilbert Subspace

There have been several attempts to generalize entanglement in high-dimensional Hilbert space. For instance, in Ref. [13] it was introduced a generalized notion of the concurrence of pure bi-partite states in $D_1 \otimes D_2$ dimension Hilbert space through the introduction of a universal inverter. According to this, they found that for $2 \otimes 3$ space, the extension of the concurrence is

$$\mathcal{C} = \sqrt{2[1 - \mathrm{tr}\left(\rho_A^2\right)]}. \tag{1}$$

Clearly, $\mathcal{C}$ measures the entanglement of a pure state in terms of the purity $\mathrm{tr}_B(\rho) = \mathrm{tr}(\rho_A^2) = \mathrm{tr}_A(\rho) = \mathrm{tr}(\rho_B^2)$ of the reduced density operators. As it is usual in standard concurrences, the qubit and the qutrit are unentangled if $\mathcal{C} = 0$ and they are maximally entangled if $\mathcal{C} = 1$. When the ground state of a system is degenerate we can employ the generalized concurrence to measure the entanglement of an individual state. If we consider the approach of [6] it is defined as the average concurrence through the ratio

$$\mathcal{C}_{av} = \frac{\int d\mu(p_1, p_2, ...)|\mathcal{C}(p_1, p_2, ...)|}{\int d\mu(p_1, p_2, ...)}, \tag{2}$$

where $d\mu(p_1, p_2, ...)$ refer to the Haar measure with respect to the parametrization $p_1, p_2, ...$, which is invariant under unitary operations. Further details concerning the above quantity are discussed in Ref. [6].

3 The Ground State of the Model

Let us consider an anisotropic Heisenberg coupling between a qubit and a qutrit in presence of a uniform magnetic field B. The coupling is described by the Hamiltonian

$$H = \frac{J}{2}\left(\sigma_x \otimes S_x + \sigma_y \otimes S_y + \eta\sigma_z \otimes S_z\right) + B\left(\frac{1}{2}\sigma_z + S_z\right), \tag{3}$$

where $\{\sigma_x, \sigma_y, \sigma_z\}$ are the Pauli matrices operating on the qubit state, $\{S_x, S_y, S_z\}$ are the respective spin operators acting on the qutrit state, J is a coupling constant, and η is the anisotropy of the coupling. In what follows we take $J = 1$. We denote by $\{|\uparrow\rangle, |\downarrow\rangle\}$ the qubit states while $\{|0\rangle, |1\rangle, |2\rangle\}$ will denote the qutrit states. In Ref. [6] it has been pointed out that the above Hamiltonian has associated six eigenvalues and six eigenstates. Furthermore, the eigenstate $|\uparrow 0\rangle$ with eigenenergy $\frac{3}{2}B + \frac{1}{2}\eta J$ and the eigenstate $|\downarrow 2\rangle$ with eigenenergy $-\frac{3}{2}B + \frac{1}{2}\eta J$ are non-entangled. The other four eigenstates with Schmidt number 2, are entangled. In the absence of a magnetic field (i.e. $B = 0$), the ground state becomes degenerate. The relation between the degeneracy and the anisotropy factor η can be described by the function $g(\eta)$ as follows [6]

$$g(\eta) = \begin{cases} 2, & \text{when } \eta < -1 \\ 4, & \text{when } \eta = -1 \\ 2, & \text{when } \eta > -1 \end{cases} \tag{4}$$

The ground state energy has a critical point $\eta_c = -1$ being 3-fold degenerate in this case. When $\eta < -1$, the two ground states $|0, \uparrow\rangle$ and $|1, \downarrow\rangle$ are degenerate spanning the 2-dimensional eigenspace

$$|\psi_1\rangle = A|0, \uparrow\rangle + B|1, \downarrow\rangle, \tag{5}$$

where $|A|^2 + |B|^2 = 1$. The generalized concurrence of this state is

$$\mathcal{C}_1(\bar{A}) = 2|AB| = 2\bar{A}\sqrt{1 - \bar{A}^2}, \tag{6}$$

where $\bar{A} = |A|$.

For $\eta > -1$, the ground state is doubly degenerate being the respective 2-dimensional eigenspace

$$\begin{aligned} |\psi_2\rangle = {} & \frac{C}{\alpha}\left(|\downarrow, 1\rangle - \frac{\eta + \sqrt{\eta^2 + 8}}{2\sqrt{2}}|\uparrow, 2\rangle\right) \\ & + \frac{D}{\beta}\left(|\downarrow, 0\rangle + \frac{\eta - \sqrt{\eta^2 + 8}}{2\sqrt{2}}|\uparrow, 1\rangle\right), \end{aligned} \tag{7}$$

where

$$\alpha = \sqrt{\left(\frac{\eta + \sqrt{8 + \eta^2}}{2\sqrt{2}}\right)^2 + 1}, \tag{8}$$

$$\beta = \sqrt{\left(\frac{\eta - \sqrt{8 + \eta^2}}{2\sqrt{2}}\right)^2 + 1} \tag{9}$$

and $|C|^2 + |D|^2 = 1$. By using (1), the generalized concurrence associated to the state (6) is [6]

$$\mathcal{C}_2(\bar{C}) = \sqrt{\frac{2[4\bar{C}^4 + 4\bar{D}^4 + \bar{C}^2\bar{D}^2(4 + \eta^2 + \eta\sqrt{8 + \eta^2})]}{8 + \eta^2}}, \tag{10}$$

where $\bar{C} = |C|$ and $\bar{D} = |D| = \sqrt{1 - \bar{C}^2}$.

4 Uncertainty of the Concurrence Between a Qubit and a Qutrit

The uncertainty of a physical quantity is defined as its standard deviation [1]. Consequently, the uncertainty of the concurrence should be

$$\Delta\mathcal{C} = \sqrt{\langle\mathcal{C}^2\rangle - \langle\mathcal{C}\rangle^2}. \tag{11}$$

As can be observed from the above equation, it is necessary first to define the average value of both $\mathcal{C}^2$ and $\mathcal{C}$. In the present work we shall assume that the average value of a quantity $f(t)$ in an interval $[t_0, t_0 + T]$ is given by

$$\langle f\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt. \tag{12}$$

Due that $0 \leq \bar{A} \leq 1$ and $0 \leq \bar{C} \leq 1$ in (6) and (10) respectively, then

$$\langle f(\bar{A})\rangle = \int_0^1 d\bar{A} f(\bar{A}), \tag{13}$$

$$\langle g(\bar{C})\rangle = \int_0^1 d\bar{C} g(\bar{C}). \tag{14}$$

It is possible to calculate the uncertainties in the concurrences $\Delta\mathcal{C}_1$ for $\eta < -1$ and $\Delta\mathcal{C}_2$ for $\eta > -1$, both as a function of the anisotropy factor, this is achieved through the use of (6) in (13) for finally by means of (11) to determine the value of $\Delta\mathcal{C}_1 = 0.074$. Analogously, Eq. (10) and (14) are used to obtain the uncertainty (11) in this case for $\Delta\mathcal{C}_2$.

The numerical results are plotted in Fig. 1. As it can be observed from such a figure, for $\eta < -1$ the concurrence between the qubit and the qutrit has a constant value. Meanwhile, for $\eta > -1$ the concurrence grows asymptotically to a value as $\eta \gg 1$.

In the present work, we make the assumption that the concurrence $\mathcal{C}$ between a qubit and a qutrit has a canonical conjugate variable, which we call π, in such a way that these two quantities satisfy an uncertainty relation of the form [1]

$$(\Delta\mathcal{C})^2(\Delta\pi)^2 \geq \left|\frac{1}{2}\langle[\mathcal{C}, \pi]\rangle\right|^2 = \left|\frac{1}{2}\langle\mathcal{C}\pi - \pi\mathcal{C}\rangle\right|^2. \tag{15}$$

Furthermore, to be $\mathcal{C}$ and π canonical conjugate variables they satisfy

$$\langle[\mathcal{C}, \pi]\rangle = ih, \tag{16}$$

where h is Planck's constant. In what follows we make $h = 1$. By substituting (16) in (15), we obtain the following uncertainty relation for the concurrence $\mathcal{C}$ between a qubit and a qutrit and its hypothetical canonical conjugate variable π

$$\Delta\mathcal{C}\Delta\pi \geq \frac{1}{2}. \tag{17}$$

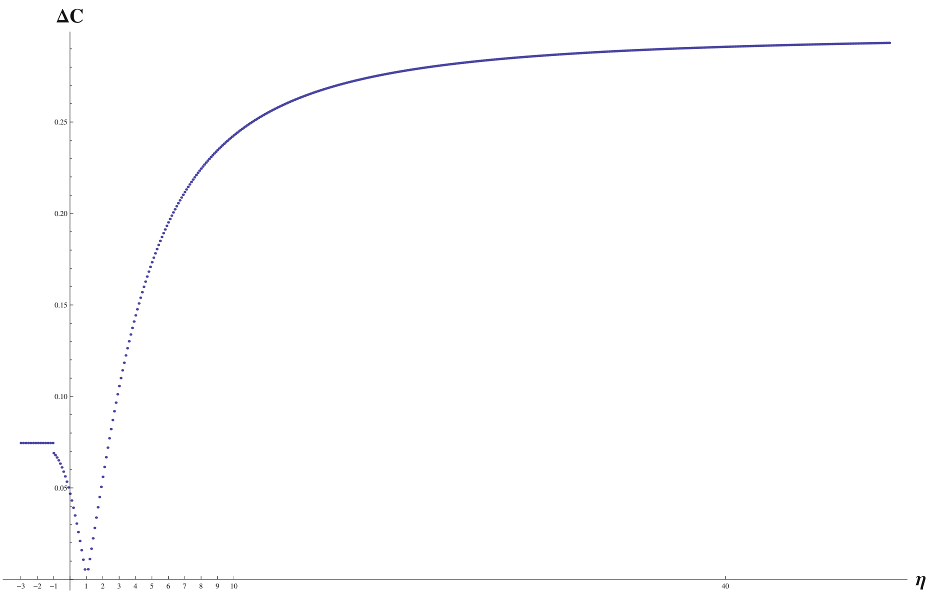

Fig. 1 Uncertainty of the concurrence between a qubit and a qutrit for both $\eta < -1$ and $\eta > -1$ as a function of the anisotropy factor η of the Hamiltonian (3)

We shall consider three different cases for the values of the anisotropy η.

(i) *Case* $\eta < -1$

As it can be observed from Fig. 1, for $\eta < -1$ the concurrence takes a constant value $\Delta\mathcal{C}_1 \sim \frac{74}{100}$. By using the above value in (17) it is concluded that for $\eta < -1$ the uncertainty for π is

$$\Delta\pi_1 \geq \frac{50}{74}. \tag{18}$$

(ii) *Case* $-1 < \eta < 1$

In this interval, the uncertainty of the concurrence is decreasing, that is, $\frac{d\mathcal{C}}{d\eta} < 0$. From (17) such behavior implies that the uncertainty for π increases notably where for $\eta = 1$ one has $\Delta(\eta = -1) = 0$ consequently there is total ignorance on the value of the variable π. In such an interval, as the value of η grows, the ignorance of the value of the canonical conjugate variable π grows too, until a total indetermination of the value of π at $\eta = -1$.

(iii) *Case* $1 < \eta$

Here the uncertainty of the concurrence increases its value as η grows, that is, $\frac{d\mathcal{C}}{d\eta} > 0$. The quantity $\Delta\mathcal{C}$ grows asymptotically to a value $\Delta\mathcal{C}_2^{\max} = \frac{1}{3}$. Consequently through the use of such a value we obtain from (17)

$$\Delta\pi_2^{\min} \geq \frac{3}{2}. \tag{19}$$

5 Conclusions

We have considered a coupling between a qubit and a qutrit interacting through an anisotropic Heisenberg Hamiltonian in the presence of a uniform magnetic field. Such a system can be implemented through the interaction of an electron with a three-level atom. The considered Hamiltonian does not depend on time. Thus, the unwelcome effects of noise are not considered. We have employed the concurrence as a measure of entanglement. It is calculated the concurrence between a qubit and a qutrit as a function of both the expansion coefficients of the composite system and the anisotropy factor η of (3). Average values of both the concurrence and its square are calculated according to (12). We consider the uncertainty of the concurrence as the standard deviation (11). The relevant point of the present work is that it is mathematically conjectured that there exists a canonical conjugate variable, which we call π, to the concurrence $\mathcal{C}$ thought of this as a dynamical variable. By hypothesis, to be π and $\mathcal{C}$ mutual canonical conjugate variables they satisfy an uncertainty relation of the form (15). We impose bounds on the uncertainty on π for several intervals of values of the anisotropy factor. With the present approach, an original point of view for the concurrence is open. It is desirable that our approach serves for encouraging searches beyond the conventional concept of concurrence. If the later is confirmed this will be of great relevance for Quantum Information Science.

Acknowledgements We acknowledge support from the PFCE 2019 resources and SNI-CONACYT for grants.

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