Universidad Autónoma del Estado de México

Facultad de Ciencias

# Algunas propiedades topológicas $\mathcal{S}_c$-preservadas y $\mathcal{S}_c$-reversibles

**TESIS POR ARTÍCULO ESPECIALIZADO**

QUE PARA OBTENER EL GRADO DE:

**Maestra en Ciencias**

PRESENTA:

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**DIRECTORES DEL TRABAJO:**
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# Resumen

El hiperespacio de todas las sucesiones convergentes no triviales de un espacio topológico Hausdorff $X$ es denotado por $\mathcal{S}_c(X)$. El objetivo principal es estudiar la preservación y la reversibilidad de una serie de propiedades topológicas que generalizan a los espacios métricos.

Este trabajo se compone de dos partes. En la primer se anexa el protocolo de investigación que se tiene registrado en Posgrado de la Facultad de Ciencias de la Universidad Autónoma del Estado de México. La segunda consta del artículo que se envió a la revista Houston Journal of Mathematics para ser evaluado, en el cual se redactaron todos los resultados de nuestra investigación.

# Protocolo

**Formato de Registro de Protocolo de Tesis de Maestría**
**(Orientación a la investigación)**

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Título de la investigación: Algunas propiedades topológicas $S_c$-preservadas y $S_c$-reversibles.

Palabras clave de la investigación (3):

| Nombre de Tutores: | Grado | LGAC del CA al que pertenece | Línea del programa académico |
|---|---|---|---|
| Tutor académico David Maya Escudero | Doctor en Ciencias | Teoría de Continuos y sus Hiperespacios | Matemáticas |
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Número de registro del proyecto de investigación del profesor asociado con el tema de la investigación:

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# Algunas propiedades topológicas $S_C$-preservadas y $S_c$-reversibles

**Resumen**. El hiperespacio de todas las sucesiones convergentes no triviales de un espacio topológico Hausdorff $X$ es denotado por $S_c(X)$. Este hiperespacio es dotado de la topología de Vietoris. El presente proyecto de investigación estuvo enfocado a estudiar la preservación y la reversibilidad de una serie de propiedades topológicas ante la operación $S_c(\cdot)$.

## 1. ANTECEDENTES

La Teoría de Hiperespacios, una rama importante de la topología, tuvo sus inicios a principios del Siglo XX y desde entonces la investigación en esta área ha experimentado un interés creciente. La teoría de hiperespacios se encarga de estudiar familias particulares de subconjuntos de los espacios topológicos. Esta teoría ha mostrado ser muy útil en el comportamiento topológico de los espacios originales con respecto a las propiedades que presentan los hiperespacios y viceversa, lo cual se refleja en la amplia bibliografía que existe al respecto. Algunos de los hiperespacios más estudiados para un espacio métrico $X$ son:

$$CL(X) = \{A \subseteq X : A \text{ es cerrado en } X \text{ y } A \neq \emptyset\},$$
$$2^X = \{A \in CL(X) : A \text{ es compacto}\},$$
$$C(X) = \{A \in 2^X : A \text{ es conexo}\},$$
$$F_n(X) = \{ A \in 2^X : A \text{ tiene a lo más } n \text{ puntos}\} \text{ y}$$
$$C_n(X) = \{ A \in 2^X : A \text{ tiene a lo más } n \text{ componentes}\}.$$

Hasta el momento, los hiperespacios más investigados han sido $CL(X)$, $2^X$ y $C(X)$. En las últimas décadas del Siglo XX hubo avances significantes en el estudio de otros hiperespacios, como $F_n(X)$ (conocido como el $n$-ésimo producto simétrico de $X$) y $C_n(X)$ (conocido como $n$-ésimo hiperespacio de $X$). Muy recientemente un nuevo hiperespacio ha atraído la atención de los especialistas, a éste se le llama *hiperespacio de sucesiones convergentes*. Dado que hasta el momento se tiene poca información sobre él, el ampliar el conocimiento fue uno de los principales objetivos de estudio del proyecto aportando al entendimiento de este novedoso hiperespacio.

## 2. JUSTIFICACIÓN

Se sabe que gran parte de las propiedades topológicas de un espacio métrico está determinada por sus sucesiones convergentes; de esta manera, resulta muy natural estudiar el comportamiento del llamado *hiperespacio de sucesiones convergentes* (no triviales), denotado por $S_c(X)$. Por esta razón, S. García-Ferreira y Y. F. Ortiz-Castillo introdujeron en su artículo [1] el hiperespacio en cuestión donde analizaron algunas de sus propiedades; ellos consideran que el estudio de $S_c(X)$ es particularmente interesante cuando consideran espacios métricos que no tienen puntos aislados. Posteriormente, en [2], [3], [4], [5] y [6] se extiende el estudio de este hiperespacio hacia los espacios Hausdorff obteniendo resultados importantes que hacen que $S_c(X)$ contraste por su comportamiento topológico con el resto de los hiperespacios.

Por otro lado, a lo largo del Siglo XX ha quedado claro que el estudio de los hiperespacios resulta sumamente útil para determinar las propiedades de los espacios en lo que éstos se basan. Los artículos publicados sobre hiperespacios se han concentrado esencialmente en el estudio de $CL(X)$, $2^X$, $C(X)$, $F_n(X)$ y $C_n(X)$; de acuerdo con esto último y al párrafo anterior, consideramos que estudiar el hiperespacio $S_c(X)$ es particularmente relevante. Este nuevo ángulo de estudio contribuye a hacer más completo el entendimiento sobre las relaciones entre los hiperespacios y los espacios base. Cabe mencionar que, hasta donde tenemos conocimiento, la información existente en la literatura sobre este hiperespacio se encuentra vertida únicamente en [1], [2], [3], [4], [5] y [6], de aquí la relevancia de este proyecto.

## 3. DEFINICIÓN DEL PROBLEMA

La convergencia de sucesiones es una importante herramienta para determinar propiedades topológicas en espacios Hausdorff. Por otra parte, el estudio de hiperespacios puede proveer de información acerca del comportamiento topológico del espacio original y viceversa. En conexión con ambos conceptos, el hiperespacio consistente de todas las sucesiones convergentes no triviales $S_c(X)$ se introduce y ha sido estudiado; donde por una sucesión convergente de un espacio Hausdorff $X$ entenderemos un subconjunto infinito numerable $S \in CL(X)$ para el cual existe $x \in S$ tal que $S \setminus U$ es finito para cada subconjunto abierto $U$ de $X$ tal que $x \in U$.

Una propiedad topológica $P$ se dice

a) $S_c$-*preservada* si siempre que un espacio Hausdorff $X$ tiene la propiedad $P$, el hiperespacio $S_c(X)$ tambien la posee.
b) $S_c$-*reversible* si la condición "$S_c(X)$ tiene la propiedad $P$" implica que $X$ tiene la propiedad $P$, para cualquier espacio Hausdorff $X$.

El presente proyecto estuvo dirigido a investigar profundamente que propiedades topológicas son $S_c$ -preservadas o $S_c$ -reversibles, preferentemente las que a continuación se enlistan: ser espacio de Lasnev, ser $\aleph_0$-espacio, ser $\alpha$-espacio, ser espacio desarrollable, ser espacio de Moore, ser $\gamma$-espacio, ser $M_1$-espacio, ser $M_2$-espacio, ser espacio de Nagata, ser $\sigma$-espacio, ser hereditariamente normal, ser perfectamente normal y ser paracompacto.

Se sabe que la conexidad es una propiedad $S_c$-preservada y $S_c$-reversible (ver [4, pp. 151, 152]), que la conexidad local es $S_c$-preservada (ver [4, p. 154]) y bajo condiciones adicionales es $S_c$-reversible (ver [4, p. 155]), la conexidad por caminos es $S_c$-reversible dentro de la clase de los espacios métricos o segundo numerables (ver [4, p. 153]) pero no es $S_c$-preservada (ver [1, p. 800]), y finalmente las propiedades de normalidad, ser espacio de Fréchet y ser secuencial no son $S_c$-preservadas (ver [4, pp. 149, 150] y [6, p. 102]).

## 4. OBJETIVOS Y METAS

1) Se determinaron cuales de las propiedades que a continuación se enlistan: ser espacio de Lasnev, ser $\aleph_0$-espacio, ser $\alpha$-espacio, ser espacio desarrollable, ser espacio de Moore, ser $\gamma$-espacio, ser $M_1$-espacio, ser $M_2$-espacio, ser espacio de Nagata, ser $\sigma$-espacio, ser hereditariamente normal, ser perfectamente normal y ser paracompacto son $S_c$-preservadas.
2) En el caso de propiedades que resultaron no ser $S_c$ -preservadas, se establecieron las condiciones necesarias o suficientes para que tal propiedad pertenezca a esta clase.
3) Se determinaron cuales de las propiedades que a continuación se enlistan: ser espacio de Lasnev, ser $\aleph_0$-espacio, ser $\alpha$-espacio, ser espacio desarrollable, ser espacio de Moore, ser $\gamma$-espacio, ser $M_1$-espacio, ser $M_2$-espacio, ser espacio de Nagata, ser $\sigma$-espacio, ser hereditariamente normal, ser perfectamente normal y ser paracompacto son $S_c$-reversibles.
4) En el caso de propiedades que resultaron no ser $S_c$-reversibles, se establecieron las condiciones necesarias o suficientes para que tal propiedad pertenezca a esta clase.

## 5. METODOLOGÍA

Se empleó la metodología usual en proyectos de investigación en matemáticas:

- Se realizaron discusiones conjuntas con los integrantes del Comité de Tutores sobre los avances conseguidos.
- Se expusieron artículos relacionados al tema de investigación ante los Tutores.
- Se realizó investigación personal.
- Se realizó investigación bibliográfica existente con fines de fortalecer la formación integral del alumno.
- Se expusieron los resultados obtenidos más importantes en congresos nacionales especializados en el área de topología.
- Se participó activamente en las sesiones semanales del Seminario Permanente de Hiperespacio de Continuos de la Facultad de Ciencias de la UAEMéx.

## 6. PRODUCTOS COMPROMETIDOS

Se envío un artículo de investigación que contiene los resultados obtenidos más importantes al proceso de arbitraje de la revista Houston Journal of Mathematics.

## 8. REFERENCIAS BIBLIOGRÁFICAS

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Considerando el año de ingreso al programa académico y el reglamento que le aplica, para el tiempo máximo a presentar la tesis o trabajo terminal de grado.

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|---|---|---|---|---|
| | 1er | 2do | 3er | 4to |
| Básica A (Análisis real y complejo) | X | | | |
| Básica B (Algebra moderna) | X | | | |
| Curso en Ciencias A | | X | | |
| Curso en Ciencias B | | X | | |
| Curso en Ciencias C | | | X | |
| Curso en Ciencias D | | | X | |
| Seminario Interdisciplinario I | X | | | |
| Seminario Interdisciplinario II | | X | | |
| Seminario Interdisciplinario III | | | X | |
| Seminario Interdisciplinario IV | | | | X |
| Actividades de Investigación de Maestría I | X | | | |
| Actividades de Investigación de Maestría II | | X | | |
| Actividades de Investigación de Maestría III | | | X | |
| Actividades de Investigación de Maestría IV | | | | X |
| Investigación conjunta con tutores académicos | X | X | X | X |
| Investigación personal | X | X | X | X |
| Investigación bibliográfica y exposición de artículos de investigación | X | X | X | X |
| Asistencia y participación en el Seminario Permanente de Teoría de Continuos e Hiperespacios | X | X | X | X |
| Exposición de los resultados obtenidos más importantes en un congreso especializado | | | X | |
| Redacción de los resultados obtenidos más importantes | | | X | X |
| Envío de artículo de investigación | | | | X |
| Obtención del grado: Se obtendrá 5 meses posteriores | | | | |

# Artículo

# SOME PRESERVED AND REVERSIBLE PROPERTIES TO THE HYPERSPACE OF CONVERGENT SEQUENCES

FÉLIX CAPULÍN, DAVID MAYA, NATALY MONDRAGÓN-CHIGORA, FERNANDO OROZCO-ZITLI

Abstract. The hyperspace of the nontrivial convergent sequences of a topological space Hausdorff $X$ is denoted by $\mathcal{S}_c(X)$. This hyperspace is endowed with the Vietoris topology. We consider several generalized metric properties and study the relation between a space $X$ satisfying such property and its hyperspace $\mathcal{S}_c(X)$ satisfying the same property.

## 1. Introduction

Convergence of sequences is an important tool to determine topological properties in Hausdorff spaces. On the other hand, the study of hyperspaces can provide information about the topological behavior of the original space and vice versa. In connection with both concepts, the hyperspace consisting of all nontrivial convergent sequences $\mathcal{S}_c(X)$, of a metric space $X$ without isolated points, was introduced in [6]. This hyperspace is endowed with the Vietoris topology. Interesting properties of this hyperspace are presented in [3, 6–8, 12–17] where the study was extended to Hausdorff spaces.

A topological property $P$ will be called:

a) $\mathcal{S}_c$-*preserved* provided if a Hausdorff space $X$ has property $P$, so does $\mathcal{S}_c(X)$, and
b) $\mathcal{S}_c$-*reversible* if the condition $\mathcal{S}_c(X)$ has property $P$ implies that $X$ has property $P$ for any Hausdorff space $X$.

An interesting problem is to determine whether a topological property is either $\mathcal{S}_c$-preserved or $\mathcal{S}_c$-reversible. In [3, 6–8, 16], the authors present the solution to this problems for several topological properties, one of the main results is that the connectedness is both $\mathcal{S}_c$-preserved and $\mathcal{S}_c$-reversible.

On the other hand, the relation between the conditions a space $X$ satisfies a generalized metric property and its hyperspace of nonempty closed subsets, its hyperspace of compact subsets, its symmetric products and its

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*Key words and phrases.* Hyperspace of nontrivial convergent sequences, Lãsnev space, $\sigma$-space, $\alpha$-space, developable space, Moore space, $\gamma$-space, Nagata space, cosmic space, $\aleph_0$-space, $G_\delta$-diagonal, $M_i$-space, hereditarily normal, perfectly normal and strongly first countable.

hyperspace of finite subsets satisfy such property is studied by several authors. Readers specially interested are referred to [9, 19–22].

In this paper, we present a partial solution to the problem of determining if several generalized metric properties are either $\mathcal{S}_c$-preserved or $\mathcal{S}_c$-reversible, particularly, we prove that being developable, being Moore and being an $\aleph_0$-space are $\mathcal{S}_c$-preserved and we exhibit examples to see being Lãsnev, being paracompact, being perfectly normal, being hereditarily normal and to have a $G_\delta$-diagonal are not $\mathcal{S}_c$-preserved. Exploring generalized metric properties that are $\mathcal{S}_c$-reversible, we find additional conditions on $X$ to prove that if $\mathcal{S}_c(X)$ is one of the following kind of spaces: cosmic, developable, Moore, an $\alpha$-space, a strongly first countable space, a $\gamma$-space, Lãsnev, paracompact, an $\aleph_0$-space, a $M_2$-space, a $M_3$-space, a $\sigma$-space or a Nagata space, then $X$ also is of the same kind space.

## 2. Preliminaries

All topological notions and all set-theoretic notions whose definition is not included here should be understood as in [5].

The symbol $\omega$ denotes both, the first infinite ordinal and the first infinite cardinal. The successor of $\omega$ is the ordinal $\omega+1=\omega\cup\{\omega\}$. The set $\omega-\{0\}$ is denoted by $\mathbb{N}$. If $X$ is a set, $[X]^{<\omega}$ is the collection of all finite subsets of $X$.

For a topological space $X$, the symbol $\tau_X$ will denote the collection of all open subsets of $X$ and, for a subset $A$ of $X$, we will use $\mathrm{Int}_X(A)$ and $\mathrm{Cl}_X(A)$ to represent its interior in $X$ and its closure in $X$, respectively.

Given a family $\{X_\alpha:\alpha\in J\}$ of pairwise disjoint topological spaces, the symbol $\bigoplus_{\alpha\in J} X_\alpha$ denotes the topological sum of $\{X_\alpha:\alpha\in J\}$ (see [5, p. 74]).

Throughout this paper, $\omega+1$ will be considered as linearly ordered topological space.

In this paper, *space* means Hausdorff space. A subset $S$ of a space $X$ will be called a *nontrivial convergent sequence* in $X$ if $S$ is countably infinite and there is $x\in S$ in such a way that $S-U\in[X]^{<\omega}$ for each open subset $U$ of $X$ with $x\in U$. When this happens, the point $x$ is called the *limit point* of $S$ and we will say that $S$ *converges* to $x$ and write $\lim S=x$. Each nontrivial convergent sequence in a space is homeomorphic to $\omega+1$.

For a space $X$, let

$$\mathcal{CL}(X)=\{A\subset X: A \text{ is closed in X and } A\neq\emptyset\},$$
$$\mathcal{K}(X)=\{A\in\mathcal{CL}(X): A \text{ is compact}\} \text{ and}$$
$$\mathcal{S}_c(X)=\{S\in\mathcal{K}(X): S \text{ is a nontrivial convergent sequence in } X\}.$$

Given a family $\mathcal{U}$ of subsets of a space $X$, we define:

$$\langle\mathcal{U}\rangle=\{A\in\mathcal{CL}(X): A\subset\bigcup\mathcal{U} \text{ and for all } U\in\mathcal{U}, A\cap U\neq\emptyset\}.$$

The *Vietoris topology* is the topology on $\mathcal{CL}(X)$ generated by the base consisting of all sets of the form $\langle\mathcal{U}\rangle$, where $\mathcal{U}\in[\tau_X]^{<\omega}$. The hyperspaces

$\mathcal{S}_c(X)$ and $\mathcal{K}(X)$ will be considered as subspaces of $\mathcal{CL}(X)$. In particular, a base for the topology of $\mathcal{S}_c(X)$ consists of all sets of the form $\langle\mathcal{U}\rangle_c = \langle\mathcal{U}\rangle \cap \mathcal{S}_c(X)$, where $\mathcal{U} \in [\tau_X]^{<\omega}$.

A collection $\mathcal{P}$ of subsets of a topological space $X$ is called a *cellular family in* $X$ if $\mathcal{P}$ is a pairwise disjoint family of nonempty open subsets of $X$. The collection of all finite cellular families of a space $X$ is denoted by $\mathfrak{C}(X)$.

The following result will be used constantly in this paper without mention it explicitly.

**Proposition 2.1** ( [16, Proposition 3.2, p. 147])**.** *For an arbitrary space* $X$*,* $\{\langle\mathcal{U}\rangle_c : \mathcal{U} \in \mathfrak{C}(X)\}$ *is a base for* $\mathcal{S}_c(X)$*.*

For subsets $A$ and $B$ of a space $X$ and $S \in \mathcal{S}_c(X)$, define

$$E(A, B, S) = \{S \cup \{a, b\} : (a, b) \in A \times B\}$$

and

$$E(A, S) = \{S \cup \{a\} : a \in A\}$$

The next result follows from [6, Lemma 1.1, p. 796] and [16, Lemma 3.4, p. 148].

**Lemma 2.2.** *Let* $X$ *be a space, let* $A, B \in \mathcal{CL}(X)$ *and let* $S \in \mathcal{S}_c(X)$*. The following statements hold.*

1) *If* $A \cap B = \emptyset$ *and* $S \cap (A \cup B) = \emptyset$*, then the closed subset* $E(A, B, S)$ *of* $\mathcal{S}_c(X)$ *is homeomorphic to* $A \times B$*.*
2) *If* $A \cap S = \emptyset$*, then the closed subset* $E(A, S)$ *of* $\mathcal{S}_c(X)$ *is homeomorphic to* $A$*.*

A topological space $X$ is *sequential* if for each nonclosed subset $A$ of $X$, there exist $\{x_n\}_{n\in\mathbb{N}}$ in $A$ and $x \in X - A$ such that $\lim x_n = x$.

For each space $X$, set $L_X = \{\lim S : S \in \mathcal{S}_c(X)\}$. If $X$ is a sequential space, then its subspace $X - L_X$ is open and discrete.

Without mention it explicitly, along of in this paper, we assume that each space $X$ satisfies that $\mathcal{S}_c(X) \neq \emptyset$.

## 3. Main Results

Let $\mathcal{A}$ be an open cover of a topological space $X$. The *star of a point* $x$ *with respect to* $\mathcal{A}$ is the set $St(x, \mathcal{A}) = \bigcup\{A \in \mathcal{A} : x \in A\}$.

A topological space $X$ is *developable* if there exists a sequence $\{\mathcal{G}_m\}_{m\in\mathbb{N}}$ of open covers of $X$ such that for each $x \in X$, the sequence $\{St(x, \mathcal{G}_m)\}_{m\in\mathbb{N}}$ is a local base of $x$.

**Theorem 3.1.** *Being developable is* $\mathcal{S}_c$*-preserved.*

*Proof.* First, observe that any subset of a developable space is a developable space. By [22, Theorem 2.7, p. 172], the hyperspace $\mathcal{K}(X)$ is developable for a developable space $X$. Thus, if $X$ is a developable space, since $\mathcal{S}_c(X)$ is contained in $\mathcal{K}(X)$, then $\mathcal{S}_c(X)$ is developable. $\square$

A topological space $X$ is *Moore* if it is regular and developable. As a consequence of Theorem 3.1 and [17, Lemma 3.1, p. 433], we obtain the next result.

**Theorem 3.2.** *Being a Moore space is $\mathcal{S}_c$-preserved.*

A space $X$ has $\mathcal{P}$ as a *network* provided that $\mathcal{P}$ is a collection of subsets of $X$ such that for each $x \in X$ and each open subset $U$ of $X$ with $x \in U$, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

A topological space $X$ is *cosmic* if $X$ has a countable network.

**Theorem 3.3.** *Being a cosmic space is not $\mathcal{S}_c$-preserved.*

*Proof.* The puropose of [17, Example 8.4, p. 440] is to exhibit a cosmic space $X$ such that each network of $\mathcal{S}_c(X)$ is uncountable. □

**Theorem 3.4.** *Being a cosmic space is $\mathcal{S}_c$-reversible.*

*Proof.* From [17, Theorem 8.3, p. 439], it follows that if $\mathcal{S}_c(X)$ is cosmic, then the space $X$ has a countable network. □

**Proposition 3.5.** *Let $X$ be a locally compact space. The following conditions are equivalent.*

(1) *$X$ is cosmic.*
(2) *$\mathcal{K}(X)$ is cosmic.*
(3) *$\mathcal{S}_c(X)$ is cosmic.*

*Proof.* Combining [5, Theorem 3.3.5, p. 149], [16, Theorem 5.5, p. 157] and [18, Theorem 4.9.12, p. 164], we get that (1) implies (2). Since being cosmic is hereditary, if $\mathcal{K}(X)$ is cosmic, then $\mathcal{S}_c(X)$ is cosmic. Finally, by Theorem 3.4, (1) is consequence of (3). □

A space $X$ has $\mathcal{P}$ as a *pseudobase* provided that $\mathcal{P}$ is a collection of subsets of $X$ satisfying for each compact subset $C$ of $X$ and for each open subset $U$ of $X$ such that $C \subset U$, there exists $P \in \mathcal{P}$ such that $C \subset P \subset U$.

A regular space $X$ is an $\aleph_0$-*space* if $X$ has a countable pseudobase.

**Theorem 3.6.** *Being an $\aleph_0$-space is $\mathcal{S}_c$-preserved.*

*Proof.* Let $X$ be an $\aleph_0$-space. By [19, Corollary 9.4, p. 993], the hyperspace $\mathcal{K}(X)$ is an $\aleph_0$-space. Since being a $\aleph_0$-space is hereditary, we have that $\mathcal{S}_c(X)$ is an $\aleph_0$-space. □

**Theorem 3.7.** *If $X$ is locally compact such that $\mathcal{S}_c(X)$ is an $\aleph_0$-space, then $X$ is an $\aleph_0$-space.*

*Proof.* According to [19, Lemma 10.1, p. 993], $\mathcal{S}_c(X)$ is cosmic. By Proposition 3.5, $\mathcal{K}(X)$ is cosmic. Thus, applying [5, Theorem 3.3.1, p. 148], the result follows from [19, Proposition 10.3, p. 995]. □

A topological space $X$ is called *Lãsnev space* if $X$ is the closed image of a metric space.

A topological space $X$ is called *paracompact space* if every open cover of $X$ has an open refinement that is locally finite.

A topological space $X$ is called *perfectly normal space* if $X$ is a normal space and every closed subset of $X$ is a $G_\delta$-set.

A topological space $X$ is called *hereditarily normal space* if every subspace of $X$ is a normal space.

Endowing with the discrete topology to the set $\{0,1\}$, for a topological space $K$, let $X_K = (K \times \{0,1\}) \bigoplus \omega + 1$. According to Lemma 2.2, the closed subspace $E(K \times \{0\}, K \times \{1\}, \omega + 1)$ of $\mathcal{S}_c(X)$ is homeomorphic to $K \times K$.

A topological property $P$ is *hereditary (hereditary with respect to closed subsets)* provided that every subspace (closed subspace) of a topological space having $P$ has $P$.

We say that a topological property $P$ is *additive* provided that the topological sum of a pairwise disjoint family of topological spaces having $P$ has $P$.

**Lemma 3.8.** *Let $P$ be an additive and hereditary with respect to closed subsets property. If $\omega + 1$ has $P$ and there exists a space $K$ having $P$ such that $K \times K$ does not have $P$, then $P$ is not $\mathcal{S}_c$-preserved.*

*Proof.* First, observe that $X_K$ has $P$. Let us prove that $\mathcal{S}_c(X_K)$ does not have $P$. Seeking a contradiction, we assume that $\mathcal{S}_c(X_K)$ has $P$. Since $P$ is hereditary with respect to closed subsets, $E(K \times \{0\}, K \times \{1\}, \omega + 1)$ has $P$ and so $K \times K$ has $P$, an absurd. Therefore, $\mathcal{S}_c(X_K)$ does not have $P$. In other words, $P$ is not $\mathcal{S}_c$-preserved. □

The sequential fan $S(\omega)$ is defined as follows. Set $S(\omega) = ((\omega + 1) \times \mathbb{N})/(\{\omega\} \times \mathbb{N})$, where $\mathbb{N}$ is considered as a discrete space. In other words, $S(\omega)$ is the quotient space obtained from $(\omega + 1) \times \mathbb{N}$ by identifying the set $\{\omega\} \times \mathbb{N}$ to a single point.

**Theorem 3.9.** *Being a Lãsnev space is not $\mathcal{S}_c$-preserved.*

*Proof.* Since $S(\omega)$ is not first countable, $S(\omega) \times S(\omega)$ is not first countable. By [11, Theorem B, p. 109], $S(\omega) \times S(\omega)$ is not Lãsnev. Notice that $\omega + 1$ is Lãsnev and that being Lãsnev is additive and hereditary. Apply Lemma 3.8 to conclude that being Lãsnev is not $\mathcal{S}_c$-preserved. □

**Theorem 3.10.** *Each one of the following properties is not $\mathcal{S}_c$-preserved.*

(1) *being a paracompact space,*
(2) *being a perfectly normal space,*
(3) *being a hereditarily normal space.*

*Proof.* Let $K$ be Sorgenfrey line (see [5, Example 1.2.2, p. 21]). The space $K$ is paracompact (see [5, Example 5.1.31, p. 309]), perfectly normal and hereditarily normal (see [5, p. 45]) such that $K \times K$ is not paracompact (see [5, Example 3.1.31, p. 309]) and $K \times K$ is neither perfectly normal nor hereditarily normal (see [5, Example 2.3.12, p. 80]). On the other hand, observe that all these properties are additive and [5, Theorems 2.1.6 and 5.1.29, pp. 68, 309] prove that each one of them is hereditary with respect to closed subsets. We apply Lemma 3.8 to complete the proof. □

Let $X$ be a space, let $\mathcal{U}$ be a subset of $\mathcal{S}_c(X)$ and let $S \in \mathcal{S}_c(X)$. Define

$$N(S,\mathcal{U}) = \{x \in X : S \cup \{x\} \in \mathcal{U}\}.$$

**Lemma 3.11.** *Let $X$ be a space and let $A$ be a non-empty subset of $X$. If $S \in \mathcal{S}_c(X)$ is such that $S \cap A = \emptyset$ and $\mathcal{U}$ is an open subset of $\mathcal{S}_c(X)$ containing $E(A,S)$, then $N(S,\mathcal{U})$ is an open subset of $X$ containing $A$.*

*Proof.* Set $U = N(S,\mathcal{U})$. To show that $U$ is an open subset of $X$, let $x \in U$. Then, $S \cup \{x\} \in \mathcal{U}$ and, we can choose $\mathcal{W} \in \mathfrak{C}(X)$ satisfying that $S \cup \{x\} \in \langle \mathcal{W} \rangle_c \subset \mathcal{U}$. Let $W_x \in \mathcal{W}$ be such that $x \in W_x$. In order to check that $W_x \subset U$, let $z \in W_x$. Since $S \cup \{x\} \subset \bigcup \mathcal{W}$ and $z \in W_x$, we deduce that $S \cup \{z\} \subset \bigcup \mathcal{W}$. Now, we take $W \in \mathcal{W}$. If $W = W_x$, then $(S \cup \{z\}) \cap W \neq \emptyset$. Next, assume that $W \neq W_x$. Since $(S \cup \{x\}) \cap W \neq \emptyset$ and $x \notin W$, we obtain that $S \cap W \neq \emptyset$. So, $(S \cup \{z\}) \cap W \neq \emptyset$. Hence $S \cup \{z\} \in \langle \mathcal{W} \rangle_c \subset \mathcal{U}$. This proves that $z \in U$. Thus, $U$ is an open subset of $X$. Finally, the assumption that $E(A,S) \subset \mathcal{U}$ guarantees that $A \subset U$. □

The proof of the next result follows from the definition of the set $E(A,S)$ and $N(S,\mathcal{U})$.

**Lemma 3.12.** *Let $X$ be a space and let $S \in \mathcal{S}_c(X)$.*

1. *If $A$ and $B$ are subsets of $X$ such that $S \cap (A \cup B) = \emptyset$, then $E(A \cap B, S) = E(A,S) \cap E(B,S)$.*
2. *If $\mathcal{U}$ and $\mathcal{W}$ are subsets of $\mathcal{S}_c(X)$, then $N(S,\mathcal{U} \cap \mathcal{W}) = N(S,\mathcal{U}) \cap N(S,\mathcal{W})$.*

**Theorem 3.13.** *Let $X$ be a space such that $L_X$ is dense. If $\mathcal{S}_c(X)$ is hereditarily normal, then $X$ is hereditarily normal.*

*Proof.* According to [5, Theorem 2.1.7, p. 69] to show that $X$ is hereditarily normal, we only need to prove that every open subspace of $X$ is normal. Let $V \in \tau_X$. Let $A, B \in \mathcal{CL}(X)$ be such that $A \cap V \neq \emptyset$, $B \cap V \neq \emptyset$ and $(A \cap V) \cap (B \cap V) = \emptyset$. We assume that $A \cup B \subsetneq X$. The fact that $L_X$ is dense implies that there exists $S \in \mathcal{S}_c(X)$ satisfying that $S \subset X - (A \cup B)$. Set $\mathcal{V} = \{X - (A \cup B), V\}$. Let

$$L_1 = E(A,S) \cap \langle \mathcal{V} \rangle_c = E(A \cap V, S) \text{ and}$$

$$L_2 = E(B,S) \cap \langle \mathcal{V} \rangle_c = E(B \cap V, S).$$

Since $E(A,S), E(B,S)$ are closed of $\mathcal{S}_c(X)$ (see Lemma 2.2), $L_1$ and $L_2$ are closed subsets of $\langle \mathcal{V} \rangle_c$. We apply Lemma 3.12 to get that $L_1 \cap L_2 = \emptyset$. Now, by the fact that $\langle \mathcal{V} \rangle_c$ is normal, there exist disjoint open subsets $\mathcal{U}, \mathcal{W}$ of $\mathcal{S}_c(X)$ such that $L_1 \subset \mathcal{U} \subset \langle \mathcal{V} \rangle_c$ and $L_2 \subset \mathcal{W} \subset \langle \mathcal{V} \rangle_c$. By lemmas 3.11 and 3.12, $N(S,\mathcal{U})$ and $N(S,\mathcal{W})$ are disjoint open subsets of $X$ such that

$A \cap V \subset N(S,\mathcal{U}) \subset V$ and $B \cap V \subset N(S,\mathcal{W}) \subset V$. Therefore, $X$ is hereditarily normal. □

Similar argument in the last proof can be used to prove the next results.

**Theorem 3.14.** *Let $X$ be a space such that $L_X$ is dense. If $\mathcal{S}_c(X)$ is normal, then $X$ is normal.*

**Theorem 3.15.** *Let $X$ be a space such that $L_X$ is dense. If $\mathcal{S}_c(X)$ is regular, then $X$ is regular.*

**Theorem 3.16.** *Let $X$ be a space such that $L_X$ is dense. If $\mathcal{S}_c(X)$ is perfectly normal, then $X$ is perfectly normal.*

*Proof.* Let $A \in \mathcal{CL}(X) - \{X\}$. The fact that $L_X$ is dense guarantees the existence of $S \in \mathcal{S}_c(X)$ such that $S \subset X - A$. According to Lemma 2.2, $E(A,S)$ is a closed subset of $\mathcal{S}_c(X)$. So, by our assumption on $\mathcal{S}_c(X)$, there exists a countable family $\{\mathcal{W}_n : n \in \mathbb{N}\}$ of open subsets of $\mathcal{S}_c(X)$ such that $E(A,S) = \bigcap_{n\in\mathbb{N}} \mathcal{W}_n$. By Lemma 3.11, $N(S,\mathcal{W}_n)$ is an open subset of $X$ containing $A$ for all $n \in \mathbb{N}$. Then, $A \subset \bigcap_{n\in\mathbb{N}} N(S,\mathcal{W}_n)$. To prove $\bigcap_{n\in\mathbb{N}} N(S,\mathcal{W}_n) \subset A$, we choose $z \in \bigcap_{n\in\mathbb{N}} N(S,\mathcal{W}_n)$. Then, $S \cup \{z\} \in \bigcap_{n\in\mathbb{N}} \mathcal{W}_n = E(A,S)$. This implies that there exists $a \in A$ such that $S \cup \{z\} = S \cup \{a\}$. The last equality and the condition $a \in A \subset X - S$ guarantee that $z = a$. Therefore, $A = \bigcap_{n\in\mathbb{N}} N(S,\mathcal{W}_n)$. We conclude that $X$ is perfectly normal. □

**Question 3.17.** Can the density of $L_X$ be omitted in theorems 3.13, 3.14, 3.15 and 3.16?

A topological property $P$ is called *strong additive* if each space written as union of two of its closed subsets having $P$ has also $P$.

**Lemma 3.18.** *Let $P$ be a strong additive and hereditary topological property. If $X$ is a space such that $|L_X| \geqslant 2$ and $\mathcal{S}_c(X)$ has $P$, then $X$ has $P$.*

*Proof.* Let $x_1, x_2 \in L_X$ be such that $x_1 \neq x_2$. Since $X$ is Hausdorff, there exist $A_1, A_2 \in \mathcal{CL}(X) - \{X\}$ such that $X = A_1 \cup A_2$, $x_1 \in X - A_2$, $x_2 \in X - A_1$. Let $S, Q \in \mathcal{S}_c(X)$ be such that $\lim S = x_1$ and $\lim Q = x_2$. We can choose $S$ and $Q$ in such a way that $S \cap A_2 = \emptyset$ and $Q \cap A_1 = \emptyset$. The assumption $P$ is hereditary guarrantees $E(S, A_2)$ and $E(Q, A_1)$ have $P$. Now, in light of Lemma 2.2, $A_1$ and $A_2$ have $P$. Finally, the fact that $P$ is strong additive implies that $X$ has $P$. □

As for definitions of $M_2$*-space*, $M_3$*-space*, *Nagata space* and $\sigma$*-space* refer to [4, pp. 106, 109] and [25, p. 57].

**Theorem 3.19.** *Let $X$ be a space such that $|L_X| \geqslant 2$. The following statements hold.*

(1) *If $\mathcal{S}_c(X)$ is a $M_2$-space, then $X$ is a $M_2$-space.*

(2) *If $\mathcal{S}_c(X)$ is a $M_3$-space, then $X$ is a $M_3$-space.*
(3) *If $\mathcal{S}_c(X)$ is a $\sigma$-space, then $X$ is a $\sigma$-space.*
(4) *If $\mathcal{S}_c(X)$ is a Nagata space, then $X$ is a Nagata space.*

*Proof.* According to [4, Theorem 2.3 and Lemma 2.7, pp. 107, 108], both being $M_2$-space and being $M_3$-space are strong additive and hereditary properties. We apply Lemma 3.18 to finish (1).

We assume that $\mathcal{S}_c(X)$ is a Nagata space. According to [4, Theorem 3.1, p. 109], $\mathcal{S}_c(X)$ is first countable and $M_3$-space. Using [16, Corollary 5.8, p. 159] and (1), we obtain that $X$ is first countable and $M_3$-space. We apply [4, Theorem 3.1, p. 109] to get that $X$ is a Nagata space.

The fact that $X$ is a $\sigma$-space is a consequence of [25, V., p. 59] and Lemma 3.18 together. □

**Question 3.20.** Can the assumption on $|L_X| \geq 2$ be removed in Theorem 3.19?

**Theorem 3.21.** *Let $X$ be a space. If $X$ is either developable or Lãsnev, then $\mathcal{S}_c(X)$ is a $\sigma$-space.*

*Proof.* Suppose that $X$ is developable. Applying Theorem 3.1, we obtain that $\mathcal{S}_c(X)$ is developable. So, by [2, Proposition 1.8, p. 603], $\mathcal{S}_c(X)$ is a $\sigma$-space.

Now, assume that $X$ is Lãsnev. By [20, Theorem 4.12, p. 92], $\mathcal{K}(X)$ is a $\sigma$-space. Therefore, using that being a $\sigma$-space is hereditary, we conclude that $\mathcal{S}_c(X)$ is a $\sigma$-space. □

**Lemma 3.22.** *Let $X$ be a space and let $A, B \in \mathcal{CL}(X)$ and let $S \in \mathcal{S}_c(X)$. If $S \cap A = \{\lim S\}$, $S \cap B = \{\lim S\}$ and $A \cap B = \{\lim S\}$, then the subset $E(A, B, S)$ of $\mathcal{S}_c(X)$ is homeomorphic to $A \times B$.*

*Proof.* Let $f : A \times B \to E(A, B, S)$ be a function given by $f(a, b) = S \cup \{a, b\}$. We observe that $f$ is bijective. In order to prove that $f$ is continuous, let $(a, b) \in A \times B$ and $\mathcal{W} \in \mathfrak{C}(X)$ satisfying that $f(a, b) \in \langle \mathcal{W} \rangle_c \cap E(A, B, S)$. Let $W_a, W_b \in \mathcal{W}$ be such that $a \in W_a, b \in W_b$. The fact that $f$ is continuous follows from the fact that $f((W_a \cap A) \times (W_b \cap B)) \subset \langle \mathcal{W} \rangle_c \cap E(A, B, S)$. Now, to check that $f^{-1}$ is continuous, we start by taking $(a, b) \in A \times B$ in such a way that $S \cup \{a, b\} \in E(A, B, S)$ and open subsets $V, W$ of $X$ such that $(a, b) \in (V \cap A) \times (W \cap B)$. We consider the following cases.

**Case I.** $\lim S \notin \{a, b\}$.

Since $X$ is Hausdorff, we can choose $\{P, Q, R\} \in \mathfrak{C}(X)$ in such a way that $S \subset P$, $a \in R \subset V \cap (X - B)$ and $b \in Q \subset W \cap (X - A)$. Set $\mathcal{U} = \{P, Q, R\}$. Then $\langle \mathcal{U} \rangle_c \cap E(A, B, S)$ is an open subset of $E(A, B, S)$ containing $S \cup \{a, b\}$. To show that $f^{-1}(\langle \mathcal{U} \rangle_c \cap E(A, B, S)) \subset (V \cap A) \times (W \cap B)$, let $(c, d) \in A \times B$ be such that $S \cup \{c, d\} \in \langle \mathcal{U} \rangle_c \cap E(A, B, S)$. The fact that $c \in A$ and $d \in B$ imply that $c \notin Q$ and $d \notin R$. So, $c \in P \cup R$ and $d \in P \cup Q$. If $c \in P$, then $(S \cup \{c, d\}) \cap R = \emptyset$, an absurd. Hence, $c \in R$ and we deduce that $c \in V \cap A$. Similarly, one can prove that $d \in W \cap B$. Thus, $(c, d) \in (V \cap A) \times (W \cap B)$.

**Case II.** $a \neq b$ and either $\lim S = a$ or $\lim S = b$ .

We suppose that $a = \lim S$. The assumption that $X$ is Hausdorff let us choose $\{P, Q, R\} \in \mathfrak{C}(X)$ in such a way that $S \cap V \subset P \subset V$ and $b \in Q \subset W \cap (X - A)$ and $S - V \subset R \subset X - (A \cup B)$. Set $\mathcal{U} = \{P, Q, R\}$. Then, $S \cup \{a, b\} \in \langle \mathcal{U} \rangle_c \cap E(A, B, S)$. To see that $f^{-1}(\langle \mathcal{U} \rangle_c \cap E(A, B, S)) \subset (V \cap A) \times (W \cap B)$, let $(c, d) \in A \times B$ be such that $S \cup \{c, d\} \in \langle \mathcal{U} \rangle_c \cap E(A, B, S)$. Since $c \in A$, $c \notin Q$, we deduce that $c \in P \subset V$. Now, if we assume that $d \in P$, we can prove that $(S \cup \{c, d\}) \cap Q = \emptyset$, an absurd. Hence, $d \in Q$ and we conclude that $(c, d) \in (V \cap A) \times (W \cap B)$.

**Case III.** $\lim S = a = b$.

Using that $X$ is Hausdorff, we can choose $\{P, Q\} \in \mathfrak{C}(X)$ in such a way that $S \cap V \cap W \subset P \subset V \cap W$ and $S - (V \cap W) \subset Q \subset X - (A \cup B)$. Set $\mathcal{U} = \{P, Q\}$. Then, $S \in \langle \mathcal{U} \rangle_c \cap E(A, B, S)$. To prove that $f^{-1}(\langle \mathcal{U} \rangle_c \cap E(A, B, S)) \subset (V \cap A) \times (W \cap B)$, let $(c, d) \in A \times B$ be such that $S \cup \{c, d\} \in \langle \mathcal{U} \rangle_c \cap E(A, B, S)$. Since $(c, d) \in A \times B$, we have that $c, d \in P \subset V \cap W$. Therefore, $(c, d) \in (V \cap A) \times (W \cap B)$.

We conclude that $f^{-1}$ is continuous. □

**Theorem 3.23.** *Let $X$ be a non-metrizable sequential space such that $|L_X| = 1$. Then there exists an infinite countable subset $\mathfrak{Z}$ of $\mathcal{S}_c(X)$ such that for every $S, Q \in \mathfrak{Z}$ with $S \neq Q$, $S \cap Q = L_X$.*

*Proof.* Let us start by proving the following claim.

**Claim**. For each $S \in \mathcal{S}_c(X)$ , there exists $Q \in \mathcal{S}_c(X)$ such that $Q \cap S = L_X$.

Since $X$ is non-metrizable, $S$ is a proper closed subset of $X$ such that $X - S$ is infinite. Set $A = X - S$. Then, $A$ is an open subset of $X$. From the inclusion $A \subset X - L_X$, it follows that $A$ is discrete. Now, if $S$ were an open subset of $X$, then [5, Proposition 2.2.4, p. 74] would imply that $X$ is the sum of metric subspace $S$ and a discrete subspace $A$ and hence $X$ would be metrizable, an absurd. Hence, $S$ is non-open subset of $X$, equivalently, $A$ is non-closed subset of $X$.

We use the assumption $X$ is sequential to find a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A$ and $x \in X - A$ such that $\lim x_n = x$. Set $Q = \{x_n\}_{n \in \mathbb{N}} \cup \{x\}$. We observe that $Q \in \mathcal{S}_c(X)$. Since $L_X \cap A = \emptyset$, we have that $S \cap Q = L_X$. This ends the proof of Claim.

Now by induction, we will prove that for each $n \in \mathbb{N}$, there exist $S_1, \ldots, S_{n+1} \in \mathcal{S}_c(X)$ such that $S_j \cap S_i = L_X$ whenever $1 \leq i < j \leq n+1$. Claim shows that for $n = 1$, the conclusion is true.

We suppose that there exist $S_1, \ldots, S_n \in \mathcal{S}_c(X)$ such that $S_j \cap S_i = L_X$ whenever $1 \leq i < j \leq n$. Set $S = \bigcup_{i=1}^{n} S_i$. Observe that $S \in \mathcal{S}_c(X)$. Finally, by Claim, there exist $S_{n+1} \in \mathcal{S}_c(X)$ such that $S \cap S_{n+1} = L_X$. In other words, there exist $S_1, \dots, S_n, S_{n+1} \in \mathcal{S}_c(X)$ such that $S_j \cap S_i = L_X$ whenever $1 \leq i < j \leq n+1$.

Then, there exists a sequence $\{\mathfrak{Z}_n\}_{n\in\mathbb{N}}$ of subsets of $\mathcal{S}_c(X)$ such that each $\mathfrak{Z}_n$ has exactly $n$ elements, if $S, Q \in \mathfrak{Z}_n$ and $S \neq Q$, then $S \cap Q = L_X$, and $\mathfrak{Z}_n \subset \mathfrak{Z}_{n+1}$ for each $n \in \mathbb{N}$. Thus, the set $\mathfrak{Z} = \bigcup_{n\in\mathbb{N}} \mathfrak{Z}_n$ satisfies the required properties. □

Let us denote by $\pi$ the natural mapping of $(\omega+1)\times\mathbb{N}$ to $S(\omega)$.

**Theorem 3.24.** *Let $X$ be a non-metrizable sequential space such that $|L_X| = 1$. Then $S(\omega)$ is embedded in $X$.*

*Proof.* Let $\mathfrak{Z}$ be as in Theorem 3.23. The fact that $\omega+1$ is homeomorphic to $S$ for all $S \in \mathcal{S}_c(X)$ guarantees that there exists an homeomorphism $h : (\omega+1)\times\mathbb{N} \to \bigoplus_{S\in\mathfrak{Z}} (S \times \{S\})$. Let $l : \bigoplus_{S\in\mathfrak{Z}} (S\times\{S\}) \to X$ be defined by $l(x, S) = x$. We will prove that $l$ is continuous. Let $W$ be an open subset of $X$. The facts that for each $S \in \mathfrak{Z}$, $l^{-1}(W) \cap (S \times \{S\}) = (S \cap W) \times \{S\}$ and $(S \cap W) \times \{S\}$ is an open subset of $S \times \{S\}$ guarantees together that $l$ is continuous.

Now, we define the function $t : S(\omega) \to X$ given by $t(x) = l(h(\pi^{-1}(x)))$. Using Transgresion Lemma (see [23, 3.22, p. 45]) and that $l$ are continuous, we deduce that $t$ is continuous. Since $t$ satisfies the conditions of [24, Lemma 2.2, p. 53], we conclude that $X$ contains a copy of $S(\omega)$. □

**Theorem 3.25.** *Let $X$ be a non-metrizable sequential space. Each one of the following conditions implies that $|L_X| \geq 2$.*

(1) *$\mathcal{S}_c(X)$ is Lãsnev.*
(2) *$\mathcal{S}_c(X)$ is strongly first countable.*
(3) *$\mathcal{S}_c(X)$ is developable.*

*Proof.* Seeking a contradiction, we suppose that $|L_X| = 1$. By Theorem 3.24, $S(\omega)$ is embedded in $X$. Let $T$ and $N$ be infinite subsets of $\mathbb{N}$ such that $T \cap N = \emptyset$ and $T \cup N = \mathbb{N} - \{1\}$, $A = \pi((\omega+1)\times T)$, $B = \pi((\omega+1)\times N)$ and $S = \pi((\omega+1)\times\{1\})$. Since $A, B, S$ satisfies the conditions of Lemma 3.22, $A \times B$ is embedded in $\mathcal{S}_c(X)$. We notice that $A \times B$ is homeomorphic to $S(\omega)\times S(\omega)$. On the other hand, the fact that $S(\omega)$ is not first countable implies that $S(\omega)\times S(\omega)$ is not first countable. Hence, neither $S(\omega)\times S(\omega)$ is developable nor $S(\omega)\times S(\omega)$ is strongly first countable, and by [11, Theorem B, p. 109], $S(\omega)\times S(\omega)$ is not Lãsnev. So, the fact that being Lãsnev, being developable and being strongly first countable are hereditary properties guarantee that if $\mathcal{S}_c(X)$ has one of these properties, then $S(\omega)\times S(\omega)$ so does, a contradiction. This ends the proof. □

A topological space $X$ *has a $G_\delta$-diagonal* if there exists a sequence $\{\mathcal{G}_m\}_{m\in\mathbb{N}}$ of open covers of $X$ such that for each $x \in X$, $\{x\} = \bigcap_{m\in\mathbb{N}} St(x, \mathcal{G}_m)$.

As for definition of *to have a $G^*_\delta$-diagonal* refer to [21, p. 300].

**Theorem 3.26.** *Let $X$ be a space. If $X$ has a $G^*_\delta$-diagonal, then $\mathcal{S}_c(X)$ has a $G_\delta$-diagonal.*

*Proof.* From the assumption $X$ has a $G^*_\delta$-diagonal and [21, Theorem 1, p. 300], it follows that $\mathcal{K}(X)$ has a $G_\delta$-diagonal. Since to have a $G_\delta$-diagonal is hereditary, we conclude that $\mathcal{S}_c(X)$ has a $G_\delta$-diagonal. □

The aim of [21, Example 3, p. 302] is to present a space $X$ having a $G_\delta$-diagonal such that $\mathcal{K}(X)$ does not have a $G_\delta$-diagonal. Using similar arguments we prove that to have a $G_\delta$-diagonal is not $\mathcal{S}_c$-preserved.

The set of all rational numbers is denoted by $\mathbb{Q}$ and set $\mathbb{I} = \mathbb{R} - \mathbb{Q}$.

**Example 3.27.** Let $X = \mathbb{R} \times (\{-1, 0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\})$. Let $p \in X$ and $\epsilon > 0$. We define $N(p, \epsilon)$ as follows.

(1) If $p = (x, \frac{1}{n})$ with $x \in \mathbb{Q}$, then

$$N(p, \epsilon) = \{p\} \cup \{(a, \frac{1}{n}) \in X : |a - x| < \epsilon, a \in \mathbb{I}\}.$$

(2) If either $p = (x, \frac{1}{n})$ or $p = (x, -1)$ with $x \in \mathbb{I}$, then

$$N(p, \epsilon) = \{p\}.$$

(3) If $p = (x, 0)$ with $x \in \mathbb{Q}$, then

$$N(p, \epsilon) = \{p\} \cup \{(a, b) \in X : 0 \leq b < |a - x| < \epsilon\}.$$

(4) If $p = (x, 0)$ with $x \in \mathbb{I}$, then

$$N(p, \epsilon) = \{p\} \cup \{(x, b) \in X : 0 < b < \epsilon\}.$$

(5) If $p = (x, -1)$ with $x \in \mathbb{Q}$, then

$$N(p, \epsilon) = \{p\} \cup \{(a, b) \in X : |a - x| < b < \epsilon\} \cup \{(c, -1) \in X : c \in \mathbb{I}, |c - x| < \epsilon\}.$$

Endow $X$ with the topology satisfying that $\{N(p, \epsilon) : \epsilon > 0\}$ is a local base at $p$ for each $p \in X$. Then, $X$ is Hausdorff. Now, for each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{N(p, \frac{1}{n})\} : p \in X\}$. Let us use the sequence $\{\mathcal{V}_n\}_{n \in \mathbb{N}}$ of open covers of $X$ to prove that $X$ has a $G_\delta$-diagonal. Let $p \in X$. We consider the following cases.

**Case I.** $p = (x, \frac{1}{k})$ with $x \in \mathbb{R}, k \in \mathbb{N}$.

The fact that $St(p, \mathcal{V}_n) \subset (x - \frac{2}{n}, x + \frac{2}{n}) \times \{\frac{1}{k}\}$ for all $n \geq k$ implies that $\bigcap_{n \in \mathbb{N}} St(p, \mathcal{V}_n) = \{p\}$.

**Case II.** $p = (x, 0)$ with $x \in \mathbb{R}$.

We observe that $St(p, \mathcal{V}_n) \subset (x - \frac{2}{n}, x + \frac{2}{n}) \times [0, \frac{1}{n})$ for all $n \in \mathbb{N}$. So, $\bigcap_{n \in \mathbb{N}} St(p, \mathcal{V}_n) = \{p\}$.

**Case III.** $p = (x, -1)$ with $x \in \mathbb{R}$.

As a consequence of that $St(p, \mathcal{V}_n) \subset (x - \frac{2}{n}, x + \frac{2}{n}) \times ((0, \frac{1}{n}) \cup \{-1\})$ for all $n \in \mathbb{N}$, we get that $\bigcap_{n \in \mathbb{N}} St(p, \mathcal{V}_n) = \{p\}$.

Therefore, $X$ has a $G_\delta$-diagonal. Thus, it only remains to show that $\mathcal{S}_c(X)$ does not have $G_\delta$-diagonal.

We suppose to the contrary that $\mathcal{S}_c(X)$ has a $G_\delta$-diagonal. Let $\{\mathfrak{V}_m\}_{m\in\mathbb{N}}$ be a sequence of open covers of $\mathcal{S}_c(X)$ such that $\{S\} = \bigcap_{m\in\mathbb{N}} St(S, \mathfrak{V}_m)$ for each $S \in \mathcal{S}_c(X)$. For each $s \in \mathbb{R}$, let $K(s) = \{(s, 0)\} \cup \{(s, \frac{1}{n}) : n \in \mathbb{N}\}$ and $L(s) = \{(s, -1)\} \cup K(s)$. Then, each $L(s) \in \mathcal{S}_c(X)$ and if $s \in \mathbb{I}$, then $K(s) \in \mathcal{S}_c(X)$. For each $s \in \mathbb{I}$, the condition $K(s) \notin \{L(s)\} = \bigcap_{n\in\mathbb{N}} St(L(s), \mathfrak{V}_n)$ allows us choose $n_s \in \mathbb{N}$ such that $K(s) \notin St(L(s), \mathfrak{V}_{n_s})$. By virtue of Second Category Theorem of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\mathrm{Int}_{\mathbb{R}}(\mathrm{Cl}_{\mathbb{R}}\{s \in \mathbb{I} : n_s = m\}) \neq \emptyset$. We will prove that there exists $t \in \mathbb{I}$ such that $K(t) \in St(L(t), \mathfrak{V}_m)$.

Choose $r \in \mathbb{Q} \cap \mathrm{Int}(\mathrm{Cl}\{s \in \mathbb{I} : n_s = m\}) \neq \emptyset$. Since $\mathfrak{V}_m$ is an open cover of $\mathcal{S}_c(X)$, there exists $\mathcal{V} \in \mathfrak{V}_m$ such that $L(r) \in \mathcal{V}$. Let $\mathcal{W} \in \mathfrak{C}(X)$ be in such a way that $L(r) \in \langle \mathcal{W} \rangle_c \subset \mathcal{V}$. Fix $\delta > 0$ satisfying that if $\lim L(r) \in W \in \mathcal{W}$, then $N(\lim L(r), \delta) \subset W$. Let $\varepsilon > 0$ be such that, for each $q \in L(r) - N(\lim L(r), \delta)$, the condition $q \in W \in \mathcal{W}$ implies that $N(q, \varepsilon) \subset W$. Set $\mathcal{U} = \{N(\lim L(r), \delta)\} \cup \{N(q, \varepsilon) : q \in L(r) - N(\lim L(r), \delta)\}$. Then, $L(r) \in \langle \mathcal{U} \rangle_c \subset \langle \mathcal{W} \rangle_c \subset \mathcal{V}$. Now, using that $r \in \mathrm{Cl}_{\mathbb{R}}\{s \in \mathbb{I} : n_s = n\}$, it can be proved that there exists $t \in \{s \in \mathbb{I} : n_s = m\} \cap (r - \min\{\epsilon, \delta\}, r + \min\{\epsilon, \delta\})$. So, $L(t), K(t) \in \langle \mathcal{U} \rangle_c \subset \mathcal{V}$. This implies that $K(t) \in St(L(t), \mathfrak{V}_m)$, a contradiction. Therefore, $\mathcal{S}_c(X)$ does not have a $G_\delta$-diagonal.

Let $X$ be a topological space and let $f, g : \mathbb{N} \times X \to \tau_X$ be functions. We consider the following properties:

$(1)_f$ $\bigcap_{n\in\mathbb{N}} f(n, x) = \{x\}$ for each $x \in X$.

$(2)_f$ $\{f(n, x)\}_{m\in\mathbb{N}}$ is a local base at $x$, for each $x \in X$.

$(3)_f$ If $y \in f(n, x)$, then $f(n, y) \subset f(n, x)$.

$(4)_f$ $f(n + 1, x) \subset f(n, x)$ for all $n \in \mathbb{N}$.

$(5)_{fg}$ If $y \in g(n, x)$, then $g(n, y) \subset f(n, x)$.

If $f$ has the properties $(1)_f$, $(3)_f$ and $(4)_f$, then $f$ is called *$\alpha$-function for $X$*. The function $f$ is called *strong function for $X$* provided that $f$ has the properties $(2)_f$, $(3)_f$ and $(4)_f$. We will say $\lceil f, g \rceil$ is a *$\gamma$-structure for $X$* if the properties $(2)_f$, $(4)_f$, $(2)_g$, $(4)_g$ and $(5)_{fg}$ are satisfied.

A topological space $X$ is an *$\alpha$-space* if $X$ has an $\alpha$-function (see [9, Lemma 3.22, p. 104]).

A topological space $X$ is an *strongly first countable* space if $X$ has an strong function (see [9, Lemma 3.24, p. 104]).

A topological space $X$ is a *$\gamma$-space* if $X$ has an $\gamma$-structure (see [9, Lemma 3.30, p. 107]).

A nonempty subset $G$ of a topological space $X$ is *discrete cellular of $X$ with respect to $\mathcal{W}(G)$* if there exists a cellular family $\mathcal{W}(G)$ such that for each $x \in G$ there exists an unique $W \in \mathcal{W}(G)$ in such a way that $G \cap W = \{x\}$.

For sake of simplicity, we only write the subset $G$ is discrete cellular in $X$ understanding that there exists the family $\mathcal{W}(G)$ satisfying the required properties.

**Proposition 3.28.** *Let $X$ be a sequential space. Then $L_X$ is discrete cellular in $X$ if and only if $S \cap L_X$ is finite for each $S \in \mathcal{S}_c(X)$.*

*Proof.* First, we suppose that $L_X$ is discrete cellular in $X$. Let $S \in \mathcal{S}_c(X)$ and set $x = \lim S$. We choose $W \in \mathcal{W}(L_X)$ such that $x \in W$. Then, $S - W$ is finite. Since $(S \cap L_X) - \{x\} \subset S - W$, we conclude that $S \cap L_X$ is finite. To prove the second part, we proceed by contradiction. We suppose that there exists $x \in L_X$ satisfying that $W \cap (L_X - \{x\}) \neq \emptyset$ for all open subset $W$ of $X$ in such a way that $x \in W$. This guarantees that $x \in \mathrm{Cl}(L_X - \{x\})$. Since $X$ is sequential, there exists $\{x_n\}_{n\in\mathbb{N}}$ in $L_X - \{x\}$ and $p \in X - (L_X - \{x\})$ such that $\lim x_n = p$. So, $S = \{p\} \cup \{x_n\}_{n\in\mathbb{N}} \in \mathcal{S}_c(X)$ and $S \subset L_X$, an absurd. Therefore, $L_X$ is discrete cellular in $X$. □

Let $X$ be a topological space and let $G$ be discrete cellular in $X$ and let $f : \mathbb{N} \times X \to \mathbb{P}(X)$ be a function. We define $\varphi_f : \mathbb{N} \times X \to \mathbb{P}(X)$ by

$$\varphi_f(n,x) = \begin{cases} f(n,x) \cap W, & \text{if } x \in G \cap W \text{ and } W \in \mathcal{W}(G), \\ \{x\}, & \text{if } x \notin G. \end{cases}$$

**Lemma 3.29.** *Let $X$ be a topological space and let $f, g : \mathbb{N} \times X \to \tau_X$ be functions. If $G$ is discrete cellular in $X$ and $X - G$ is discrete, then the image of $\varphi_f$ and the image of $\varphi_g$ are contained in $\tau_X$ and each one of the following statements are true:*

1. *If $f$ has $(1)_f$, then $\varphi_f$ has $(1)_{\varphi_f}$.*
2. *If $f$ has $(2)_f$, then $\varphi_f$ has $(2)_{\varphi_f}$.*
3. *$\varphi_f$ has $(3)_{\varphi_f}$.*
4. *If $f$ has $(4)_f$, then $\varphi_f$ has $(4)_{\varphi_f}$.*
5. *If $g(n,x) \subset f(n,x)$ for each $(n,x) \in \mathbb{N}\times X$, then $\varphi_f, \varphi_g$ have $(5)_{\varphi_f \varphi_g}$.*

*Proof.* First, since $\mathcal{W}(G)$ is a cellular family and $X - G$ is discrete, $\varphi_f$ and $\varphi_g$ are well defined and the image of each one of them is contained in $\tau_X$.

In order to show (1), let $x \in X$. We have that $x \in \varphi_f(n,x) \subset f(n,x)$ for each $n \in \mathbb{N}$ . Then $x \in \bigcap_{n\in\mathbb{N}} \varphi_f(n,x) = \bigcap_{n\in\mathbb{N}} f(n,x) = \{x\}$. This implies that $\bigcap_{n\in\mathbb{N}} \varphi_f(n,x) = \{x\}$. Therefore, $a)$ is true.

Now, we suppose that $\{f(n,x)\}_{n\in\mathbb{N}}$ is a local base at $x$. Let $U$ be an open subset of $X$ in such a way that $x \in U$. We choose $n \in \mathbb{N}$ such that $f(n,x) \subset U$. Then, $x \in \varphi_f(n,x) \subset f(n,x) \subset U$. This proves (2).

To prove (3), assume $y \in \varphi_f(n,x)$. If $x \notin G$, then $y \in \varphi_f(n,x) = \{x\}$ and $\varphi_f(n,y) = \varphi_f(n,x)$. The condition $y \notin G$ implies that $\{y\} = \varphi_f(n,y) \subset \varphi_f(n,x)$. Now, suppose that $x, y \in G$. Then, there exists $W \in \mathcal{W}(G)$ such that $x \in W$. The inclusion $\varphi_f(n,x) \subset f(n,x)$ guarantees that $y \in f(n,x)$.

So, $y \in \varphi_f(n,x) = f(n,x) \cap W$. Hence, $y \in G \cap W = \{x\}$. Thus, $x = y$ and $\varphi_f(n,y) = \varphi_f(n,x)$.

Let $n \in \mathbb{N}$. We will show that $\varphi_f(n+1,x) \subset \varphi_f(n,x)$. Since $f$ has $(4)_f$, $f(n+1,x) \subset f(n,x)$. If $x \in G \cap W$ for some $W \in \mathcal{W}(G)$, then $\varphi_f(n+1,x) = f(n+1,x) \cap W \subset f(n,x) \cap W = \varphi_f(n,x)$. Otherwise, $\varphi_f(n+1,x) = \{x\} = \varphi_f(n,x)$. This completes the proof of (4).

We will see that (5) is true. Assume that $y \in \varphi_g(n,x)$. Under the assumption $x \notin G$, we have that $y \in \varphi_g(n,x) = \{x\}$ and so $\varphi_g(n,y) = \varphi_g(n,x)$. If $y \notin G$, then $\{y\} = \varphi_g(n,y) \subset \varphi_g(n,x)$. Now, suppose that $x, y \in G$. Let $W \in \mathcal{W}(G)$ be such that $x \in W$. Then, $y \in \varphi_g(n,x) = g(n,x) \cap W$. This implies that $y \in W \cap G = \{x\}$. So, $\varphi_g(n,y) = \varphi_g(n,x) \subset g(n,x) \cap W \subset f(n,x) \cap W = \varphi_f(n,x)$. □

The next result is a consequence of [18, 2.3.1 of Lemma 2.3, p. 156].

**Lemma 3.30.** *Let $X$ be a space and let $\mathcal{U}$ and $\mathcal{V} \in \mathfrak{C}(X)$. If $\bigcup \mathcal{U} \subset \bigcup \mathcal{V}$ and for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $U \subset V$, then $\langle \mathcal{U} \rangle_c \subset \langle \mathcal{V} \rangle_c$.*

Given a space $X$, a function $f : \mathbb{N} \times X \longrightarrow \mathbb{P}(X)$ and $S \in \mathcal{S}_c(X)$, we define the set

$$D_f(n,S) = S - \bigcup \{\varphi_f(n,y) : y \in S \cap L_X\}$$

and the function $\psi_f : \mathbb{N} \times \mathcal{S}_c(X) \to \mathbb{P}(\mathcal{S}_c(X))$ by $\psi_f(n,S) = \langle \mathcal{V}_f(n,S) \rangle$, where

$$\mathcal{V}_f(n,S) = \{\varphi_f(n,y) : y \in (S \cap L_X) \cup D_f(n,S)\}.$$

**Lemma 3.31.** *Let $X$ be a sequential space such that $L_X$ is discrete cellular in $X$ and let $f : \mathbb{N} \times X \to \tau_X$ be a function. Each one of the following statements holds:*

1. *If $x, y \in L_X$ with $x \neq y$ and $n \in \mathbb{N}$, then $\varphi_f(n,x) \cap \varphi_f(n,y) = \emptyset$.*
2. *If $S \in \mathcal{S}_c(X)$ and $f$ has $(1)_f$, then $\bigcap_{n\in\mathbb{N}} (\bigcup_{y \in S \cap L_X} \varphi_f(n,y)) = S \cap L_X$.*
3. *$\psi_f(\mathbb{N} \times \mathcal{S}_c(X)) \subset \tau_{\mathcal{S}_c(X)}$.*
4. *If $S \in \mathcal{S}_c(X)$ and $Q \in \psi_f(n,S)$, then $Q \cap L_X \subset S \cap L_X$.*
5. *If $S \in \mathcal{S}_c(X)$ and $Q \in \psi_f(n,S)$, then $\bigcup \mathcal{V}_f(n,Q) \subset \bigcup \mathcal{V}_f(n,S)$.*
6. *$\psi_f$ has $(3)_{\psi_f}$*
7. *If $f$ has $(4)_f$, then $\psi_f$ has $(4)_{\psi_f}$.*

*Proof.* Let us see (1) is true. If $W_x, W_y \in \mathcal{W}(L_X)$ fulfill that $x \in W_x$ and $y \in W_y$, then the cellularity of $\mathcal{W}(L_X)$ and the fact that $\varphi_f(n,x) \subset W_x$ and $\varphi_f(n,y) \subset W_y$ imply that $\varphi_f(n,x) \cap \varphi_f(n,y) = \emptyset$.

To check (2), notice that $S \cap L_X \subset \bigcap_{n\in\mathbb{N}} (\bigcup_{y \in S \cap L_X} \varphi_f(n,y))$. Now, let $w \in \bigcap_{n\in\mathbb{N}} (\bigcup_{y \in S \cap L_X} \varphi_f(n,y))$ and fix $n \in \mathbb{N}$. Since $w \in \bigcup_{y \in S \cap L_X} \varphi_f(n,y)$, there exists $p \in S \cap L_X$ in such a way that $w \in \varphi_f(n,p)$. Now, using (1), we obtain that $w \notin \varphi_f(n,z)$ for all $z \in S \cap L_X - \{p\}$. So, the fact that $w \in$

$\bigcap_{n\in\mathbb{N}}(\bigcup_{y\in S\cap L_X}\varphi_f(n,y))$ implies that $w\in\bigcap_{n\in\mathbb{N}}\varphi_f(n,p)$. Finally, applying (1) of Lemma 3.29, we conclude that $w=p$. Hence, $\bigcap_{n\in\mathbb{N}}(\bigcup_{y\in S\cap L_X}\varphi_f(n,y))\subset S\cap L_X$ and this finishes the proof of (2).

In order to verify that $\psi_f(\mathbb{N}\times\mathcal{S}_c(X))\subset\tau_{\mathcal{S}_c(X)}$, let $(n,S)\in\mathbb{N}\times\mathcal{S}_c(X)$. By Proposition 3.28 and Lemma 3.29, $S\cap L_X$ is finite and $\varphi_f(\mathbb{N}\times X)\subset\tau_X$. So, $\{\varphi_f(n,y):y\in S\cap L_X\}\in[\tau_X]^{<\omega}$. Now, since $\bigcup_{z\in S\cap L_X}\varphi_f(n,z)$ is an open subset of $X$ containing $\lim S$, $D_f(n,S)$ is finite. Hence, $\{\varphi_f(n,y):y\in D_f(n,S)\}\in[\tau_X]^{<\omega}$. Therefore, $\mathcal{V}_f(n,S)\in[\tau_X]^{<\omega}$ and we conclude that $\psi_f(n,S)\in\tau_{\mathcal{S}_c(X)}$.

To show (4), let $y\in Q\cap L_X$. Since $Q\subset\bigcup\mathcal{V}_f(n,S)$ and $D_f(n,S)\subset X-L_X$, there exists $x\in S\cap L_X$ such that $y\in\varphi_f(n,x)$. Therefore, using (1), we deduce that $x=y$ and this means that $y\in S\cap L_X$.

Now, we assume that $Q\in\psi_f(n,S)$. From (4), we obtain that $\{\varphi_f(n,x):x\in Q\cap L_X\}\subset\{\varphi_f(n,x):x\in S\cap L_X\}\subset\mathcal{V}_f(n,S)$. On other hand, we notice that $\bigcup\{\varphi_f(n,x):x\in D_f(n,Q)\}=\bigcup\{\{x\}:x\in D_f(n,Q)\}\subset Q\subset\bigcup\mathcal{V}_f(n,S)$. Thus, $\bigcup\mathcal{V}_f(n,Q)\subset\bigcup\mathcal{V}_f(n,S)$ and (5) is satisfied.

To check (6), let $R\in\psi_f(n,Q)$. By (5), we have that $R\subset\bigcup\mathcal{V}_f(n,S)$. Let $y\in(S\cap L_X)\cup D_f(n,S)$. To see that $R\cap\varphi_f(n,y)\neq\emptyset$, we consider three cases.

**Case I.** $y\in Q\cap L_X$.

The fact that $R\cap\varphi_f(n,y)\neq\emptyset$ is a consequence of that $R\in\psi_f(n,Q)$.

**Case II.** $y\in(S\cap L_X)-(Q\cap L_X)$.

Since $Q\cap\varphi_f(n,y)\neq\emptyset$, we take $q\in Q\cap\varphi_f(n,y)$. If $q$ were an element of $Q\cap L_X$, then $q$ would be an element of $\varphi_f(n,y)\cap\varphi_f(n,q)$, this contradicts (1). Hence, $q\in D_f(n,Q)$. Since $R\cap\varphi_f(n,q)\neq\emptyset$ and $\varphi_f(n,q)=\{q\}$, we conclude that $q\in R$. Therefore, $q\in R\cap\varphi_f(n,y)$.

**Case III.** $y\in D_f(n,S)$.

Since $\varphi_f(n,y)=\{y\}$ and $Q\cap\varphi_f(n,y)\neq\emptyset$, $y\in Q$. By (4), $\bigcup_{x\in Q\cap L_X}\varphi_f(n,x)\subset\bigcup_{x\in S\cap L_X}\varphi_f(n,x)$ and from the fact that $Q\in\langle\mathcal{V}(n,S)\rangle_c$, it follows that $D_f(n,S)=S-\bigcup_{x\in S\cap L_X}\psi_f(n,x)\subset Q-\bigcup_{x\in Q\cap L_X}\psi_f(n,x)=D_f(n,Q)$. So, $y\in D_f(n,Q)$ and the fact that $R\cap\varphi_f(n,y)\neq\emptyset$ guarantee that $y\in R$. In conclusion, $R\cap\varphi_f(n,y)\neq\emptyset$.

Therefore, $R\in\psi_f(n,S)$ and this finishes the proof of (6).

In order to show (7), we will employ Lemma 3.30. Let $n\in\mathbb{N}$. Notice that $D_f(n+1,S)\subset D_f(n,S)\cup\{\varphi_f(n,y):y\in S\cap L_X\}$ and $\varphi_f(n+1,y)\subset\varphi_f(n,y)$ for all $y\in X$. So, $\bigcup\mathcal{V}_f(n+1,S)\subset\bigcup\mathcal{V}_f(n,S)$. To end the proof we will see that each member of $\mathcal{V}_f(n,S)$ contains an element of $\mathcal{V}_f(n+1,S)$. Let $y \in (S \cap L_X) \cup D_f(n,S)$. If $y \in S \cap L_X$, then the result follows from $\varphi_f(n+1,y) \subset \varphi_f(n,y)$ and $\varphi_f(n+1,y) \in \mathcal{V}_f(n+1,S)$. Now, we assume that $y \in D_f(n,S)$. We apply (4) of Lemma 3.29 to conclude that $D_f(n,S) \subset D_f(n+1,S)$. So, $y \in D_f(n+1,S)$. The result follows that $\varphi_f(n,y) = \{y\} = \varphi_f(n+1,y)$ and $\varphi_f(n+1,y) \in \mathcal{V}_f(n+1,S)$. This finishes (7). □

**Theorem 3.32.** *Let $X$ be a sequential space such that $L_X$ is discrete cellular in $X$. If $X$ is an $\alpha$-space, then $\mathcal{S}_c(X)$ is an $\alpha$-space.*

*Proof.* Let $f$ be an $\alpha$-function for $X$. We will prove that $\psi_f$ is an $\alpha$-function for $\mathcal{S}_c(X)$. According to (3) of Lemma 3.31, $\psi_f$ is well defined. The properties $(3)_{\psi_f}$ and $(4)_{\psi_f}$ follow from (6) and (7) of Lemma 3.31, respectively. It only remains to prove that $\psi_f$ has $(1)_{\psi_f}$.

Let $S \in \mathcal{S}_c(X)$. We will see that $\bigcap_{n\in\mathbb{N}} \psi_f(n,S) \subset \{S\}$. Let $Q \in \bigcap_{n\in\mathbb{N}} \psi_f(n,S)$. To show that $Q \subset S$, let $q \in Q$. The fact that $Q \in \bigcap_{n\in\mathbb{N}} \psi_f(n,S)$ implies that $q \in \bigcap_{n\in\mathbb{N}} \left( \bigcup_{y\in(S\cap L_X)\cup D_f(n,S)} \varphi_f(n,y) \right)$. When $q \in \bigcap_{n\in\mathbb{N}} \left( \bigcup_{y\in S\cap L_X} \varphi_f(n,y) \right)$, (2) of Lemma 3.31 guarantees that $q \in S$. On the other hand, if there exist $n \in \mathbb{N}$ and $y \in D_f(n,S)$ such that $q \in \varphi_f(n,y)$, using that $\varphi_f(n,y) = \{y\}$, we conclude that $q \in S$. Hence, $Q \subset S$. Now, to show that $S \subset Q$, let $x \in S$. If $\lim Q = x$, then $x \in Q$. Otherwise, we analyse the following cases.

**Case I.** $x \notin L_X$.

By (1) of Lemma 3.29, $\bigcap_{n\in\mathbb{N}} \varphi_f(n,y) = \{y\}$ for all $y \in S \cap L_X$. Since $x \notin S \cap L_X$, using that $\varphi_f$ has $(4)_{\varphi_f}$ and the fact that $S \cap L_X \in [X]^{<\omega}$, we can choose $m \in \mathbb{N}$ such that $x \notin \varphi_f(m,y)$ for all $y \in S \cap L_X$. So, $x \in D_f(m,S)$. We infer that $\varphi_f(m,x) = \{x\}$. Since $Q \cap \varphi_f(m,x) \neq \emptyset$, we conclude that $x \in Q$.

**Case II.** $x \in L_X - \{\lim Q\}$.

Set $F = Q - \varphi_f(1,\lim Q)$. Since $\varphi_f(1,\lim Q) \in \tau_X$, $F$ is finite. From (1) and (4) of Lemma 3.31, it follows that $\varphi_f(n,x) \cap \varphi_f(1,\lim Q) = \emptyset$. Now, the fact that $Q \in \bigcap_{n\in\mathbb{N}} \psi_f(n,S)$ implies that $\in Q \cap \varphi_f(n,x) \neq \emptyset$. Thus, $F \cap \varphi_f(n,x) \neq \emptyset$ for each $n \in \mathbb{N}$. Suppose that $x \notin F$. By (1) of Lemma 3.29, $\bigcap_{n\in\mathbb{N}} \varphi_f(n,x) = \{x\}$. Then, the fact that $\varphi_f$ has $(4)_{\varphi_f}$ allows us choose $m \in \mathbb{N}$ satisfying that $F \cap \varphi_f(m,x) = \emptyset$, a contradiction. This proves that $x \in F$. Hence, $S \subset Q$.

Therefore, $Q = S$ and we conclude that $\bigcap_{n\in\mathbb{N}} \psi_f(n,S) \subset \{S\}$. In conclusion $\psi_f$ has $(1)_{\psi_f}$ □

The next example shows that the assumption on $L_X$ can be omitted in the last result, this means that there exists a sequential space $X$ such that $X$ is an $\alpha$-space, $L_X$ is not discrete cellular in $X$ and $\mathcal{S}_c(X)$ is an $\alpha$-space.

We will use the symbol $[k,\omega]$ to denote the set $\{l \in \omega : l \geq k\} \cup \{\omega\}$ and the symbol $\omega^\omega$ to represent the set of all functions from $\omega$ into $\omega$.

**Example 3.33.** Let $x \notin (\omega+1) \times \omega$. We define $X = \{x\} \cup (\omega+1) \times \omega$. Let $S(k,m) = \{[k,\omega] \times \{m\} : (k,m) \in \omega \times \omega\}$. For each $\beta \in \omega^\omega$ and $m \in \omega$, set $L(\beta,m) = \bigcup\{S(\beta(l),l) : l \geq m\} \cup \{x\}$. Let $X$ be topologized as follows. Each point in $\omega \times \omega$ is isolated. For $m \in \omega$, $\{S(k,m) : k \in \mathbb{N}\}$ is a system neighborhood in $X$ at the point $(\omega,m)$. The collection $\{L(\beta,m) : \beta \in \omega^\omega, m \in \mathbb{N}\}$ is a system neighborhood in $X$ at the point $x$. The space obtained is called *Arens space*.

First, let us argue that $X$ is an $\alpha$-space. For each $n \in \omega$, let $\beta_n \in \omega^\omega$ be defined by $\beta_n(l) = n$. Define $f : \omega \times X \to \tau_X$ by

$$f(n,y) = \begin{cases} L(\beta_n(l),n), & \text{if } y = x, \\ S(n,m), & \text{if } y = (\omega,m) \in \{\omega\} \times \omega, \\ \{y\}, & \text{if } y \in \omega \times \omega \end{cases}$$

to get an $\alpha$-function for $X$. For $(n,S) \in \mathbb{N} \times \mathcal{S}_c(X)$, let $D(n,S) = \{x\} \cup ((S \cap L_X) - f(n,x))$. We define the function $\psi : \mathbb{N} \times \mathcal{S}_c(X) \to \tau_{\mathcal{S}_c(X)}$ by $\psi(n,S) = \langle \mathcal{V}(n,S) \rangle_c$, where

$$\mathcal{V}(n,S) = \begin{cases} \{f(n,y) : y \in D(n,S) \cup (S - \bigcup_{z \in D(n,S)} f(n,z)\}, & \text{if } x \in S, \\ \{f(n,y) : y \in (S \cap L_X) \cup (S - \bigcup_{z \in S \cap L_X} f(n,z))\}, & \text{if } x \notin S. \end{cases}$$

We observe that $f|_{X-\{x\}}$ is an $\alpha$-function for $X - \{x\}$ and $L_{X-\{x\}}$ is discrete cellular in $X - \{x\}$. Then $\psi|_{\langle X-\{x\}\rangle_c} = \psi_{f|_{X-\{x\}}}$ is an $\alpha$-function for the subspace $\langle\{X - \{x\}\}\rangle_c$ of $\mathcal{S}_c(X)$. Thus, to prove that $\psi$ is an $\alpha$-function, it suffices to show that $\varphi(n,S) \in \tau_{\mathcal{S}_c(X)}$ and the conditions $(1)_\varphi, (3)_\varphi, (4)_\varphi$ are true for each $S \in \mathcal{S}_c(X)$ such that $x \in S$. To this end, we shall argue some claims.

Now, let $S \in \mathcal{S}_c(X)$ be such that $x \in S$.

**Claim 1.** $\psi(n,S) \in \tau_{\mathcal{S}_c(X)}$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Since $\lim S \in \bigcup_{z \in D(n,S)} f(n,z)$, $S - \bigcup_{z \in D(n,S)} f(n,z)$ is finite. Also, $S \cap L_X - f(n,x) \subseteq \{(\omega,m) : m \leq n\}$ is finite. Then, $D(n,S)$ is finite. Therefore, $\mathcal{V}(n,S) \in [\tau_X]^{<\omega}$ and we conclude that $\psi(n,S) \in \tau_{\mathcal{S}_c(X)}$.

**Claim 2.** If $q \in f(n,p)$, then $f(n,q) \subset f(n,p)$.

The claim follows from the definition of $f$.

**Claim 3.** If $Q \in \psi(n,S)$, then $\psi(n,Q) \subset \psi(n,S)$.

We will use Lemma 3.30. First, we assume $x \in Q$. We notice that $(Q \cap L_X) - f(n,x) \subset (S \cap L_X) - f(n,x)$ and $Q - \bigcup_{z \in D(n,Q)} f(n,z) \subset (S - \bigcup_{z \in D(n,S)} f(n,z)) \cup \{f(n,y) : y \in (S \cap L_X) - f(n,x)\}$. Hence, $\bigcup \mathcal{V}(n,Q) \subset \bigcup \mathcal{V}(n,S)$. Now, let $y \in D(n,S) \cup (S - \bigcup_{z \in D(n,S)} f(n,z))$. Then, there exists $w \in D(n,Q) \cup (Q - \bigcup_{z \in D(n,Q)} f(n,z))$ such that $w \in f(n,y)$. By Claim 2, $f(n,w) \subset f(n,y)$. Thus, $\psi(n,Q) \subset \psi(n,S)$. Next, suppose that $x \notin Q$. We have that $Q \cap L_X \subset (S \cap L_X) \cup f(n,x)$ and $Q - \bigcup_{z \in Q \cap L_X} f(n,z) \subset (S - \bigcup_{z \in D(n,S)} f(n,z)) \cup \{f(n,y) : y \in D(n,S)\}$. Then, $\bigcup \mathcal{V}(n,Q) \subset \bigcup \mathcal{V}(n,S)$. Finally, let $y \in D(n,S) \cup (S - \bigcup_{z \in D(n,S)} f(n,z))$. Thus, there exists $w \in (Q \cap L_X) \cup (Q - \bigcup_{z \in Q \cap L_X} f(n,z))$ such that $w \in f(n,y)$. The inclusion $f(n,w) \subset f(n,y)$ follows from Claim 2. Therefore, $\psi(n,Q) \subset \psi(n,S)$.

**Claim 4.** $\bigcap_{n \in \mathbb{N}} \psi(n,S) \subset \{S\}$.

Let $Q \in \bigcap_{n \in \mathbb{N}} \psi(n,S)$. To prove that $Q \subset S$, let $q \in Q$. Since $Q \in \bigcap_{n \in \mathbb{N}} \psi(n,S)$, for each $n \in \mathbb{N}$, there exists $y_n \in D(n,S) \cup (S - \bigcup_{z \in D(n,S)} f(n,z))$ such that $q \in f(n,y_n)$. If $y_n = x$ for all $n \in \mathbb{N}$, then $q \in \bigcap_{n \in \mathbb{N}} f(n,x) = \{x\}$ and we deduce that $q \in S$. Now, if there exists $m \in \mathbb{N}$ such that $y_m \notin D(m,S)$, then $y_m \in S - \bigcup_{z \in D(m,S)} f(m,z)$ and $q \in f(m,y_m) = \{y_m\} \subset S$. Next, assume that each $y_n \in (S \cap L_X) - f(n,x)$. From the fact that $q \in f(n,y_n)$ for each $n \in \mathbb{N}$, it follows that $(y_n)_{n \in \mathbb{N}}$ is a constant sequence. Hence, $q \in \bigcap_{n \in \mathbb{N}} f(n,y_1) = \{y_1\}$. This implies that $q = y_1 \in S$.

In order to prove that $S - \{x\} \subset Q$, let $y \in S - \{x\}$. Assume $y \in \omega \times \omega$. Then, there exists $k \in \mathbb{N}$ such that $y \in S - \bigcup_{z \in D(k,S)} f(k,z)$. Since $Q \in \psi(k,S)$ and $f(k,y) \in \mathcal{V}(k,S)$, $Q \cap f(k,y) \neq \emptyset$. From the fact that $f(k,y) = \{y\}$, it follows that $y \in Q$. Now, suppose that $y \in L_X$. In the case that $\lim Q = y$, we have that $y \in Q$. Assume that $y \neq \lim Q$ and $y = (\omega, l)$. Hence, $Q \cap ((\omega+1) \times \{l\})$ must be finite. Then, there exists $k \in \mathbb{N}$ such that $y \notin f(k,x)$ and $f(k,y) \cap Q \cap (\omega \times \{l\})$ is empty. So, since $f(k,y) \in \mathcal{V}(k,S)$ and $Q \in \psi(k,S)$, $f(k,y) \cap Q = \{y\}$. In conclusion, $y \in Q$. Finally, we will check that $x \in Q$. We suppose that $\lim Q \neq x$ and $\lim Q = (\omega, r)$. From the fact that $(\omega+1) \times \{r\}$ is an open subset of $X$ and $\lim Q \in (\omega+1) \times \{r\}$, the set $Q - (\omega+1) \times \{r\}$ is finite and so there exists $k \in \mathbb{N}$ such that $\lim Q \notin f(k,x)$ and $f(k,x) \cap Q \cap ((\omega+1) \times \omega)$ is empty. Finally, the conditions $f(k,x) \in \mathcal{V}(k,S)$ and $Q \in \psi(k,S)$ together guarantees that the unique point contained in $f(k,x)\cap Q$ is $x$. In conclusion, $x\in Q$. Therefore, $S\subset Q$.

**Claim 4.** $\psi(n+1,S)\subset\psi(n,S)$.

We observe that $\{f(n+1,y): y\in D(n+1,S)\}\subset\{f(n,y): y\in D(n,S)\}$ and $\{f(n+1,y): y\in S-\bigcup_{z\in D(n+1,S)} f(n,z)\}\subset\{f(n,y): y\in (S-\bigcup_{z\in D(n,S)} f(n,z))\cup D(n,S)\}$. Hence, $\bigcup\mathcal{V}(n+1,S)\subset\bigcup\mathcal{V}(n,S)$. Now, let $y\in D(n,S)\cup(S-\bigcup_{z\in D(n,S)} f(n,z))$. Then, there exists $w\in D(n+1,S)\cup(S-\bigcup_{z\in D(n+1,S)} f(n,z))$ such that $w\in f(n,y)$. By Claim 2 and the condition $f(n+1,w)\subset f(n,w)$, we have that $f(n+1,w)\subset f(n,y)$. Then, $\psi(n+1,S)\subset\psi(n,S)$.

Therefore, $\mathcal{S}_c(X)$ is an $\alpha$-space.

As for definition of $\omega\Delta$-space refer to [10, p. 641]

**Theorem 3.34.** *Let $X$ be a regular $\omega\Delta$-space. If $X$ is an $\alpha$-space, then $\mathcal{S}_c(X)$ is an $\alpha$-space.*

*Proof.* According to [10, Theorem 4.6, p. 649], $X$ is Moore. By Theorem 3.2, $\mathcal{S}_c(X)$ is Moore. Hence, since any Moore space is an $\alpha$-space (see [10, p. 651]), $\mathcal{S}_c(X)$ is an $\alpha$-space. □

**Theorem 3.35.** *Let $X$ be a sequential space such that $L_X$ is discrete cellular in $X$. If $X$ is a strongly first countable space, then $\mathcal{S}_c(X)$ is a strongly first countable space.*

*Proof.* Let $f$ be a strong function for $X$. We will prove that $\psi_f$ is a strong function for $\mathcal{S}_c(X)$. By (3) of Lemma 3.31, $\psi_f$ is well defined. The properties $(3)_{\psi_f}$ and $(4)_{\psi_f}$ follow from (6) and (7) of Lemma 3.31, respectively. Let us prove that $\psi_f$ has $(2)_{\psi_f}$.

Let $S\in\mathcal{S}_c(X)$. To show that $\{\psi_f(n,S)\}_{n\in\mathbb{N}}$ is a local base at $S$, we choose $\mathcal{U}\in\mathfrak{C}(X)$ satisfying that $S\in\langle\mathcal{U}\rangle_c$. For each $y\in S\cap L_X$, let $U_y\in\mathcal{U}$ be such that $y\in U_y$.

Using that $S\cap L_X\in[X]^{<\omega}$ and the fact that $\varphi_f$ has $(4)_{\varphi_f}$, we can choose $m\in\mathbb{N}$ such that $\varphi_f(m,y)\subset U_y$ for all $y\in S\cap L_X$. We will show that $\psi_f(m,S)\subset\langle\mathcal{U}\rangle_c$. Let $R\in\psi_f(m,S)$. We start by taking $U\in\mathcal{U}$ and we will show that $R\cap U\neq\emptyset$. When $U=U_y$ for some $y\in S\cap L_X$, the fact that $R\cap\varphi_f(m,y)\neq\emptyset$ implies that $R\cap U_y\neq\emptyset$. Otherwise, since $S\in\langle\mathcal{U}\rangle_c$, we can choose $x\in U\cap D_f(m,S)$. So, using that $\varphi_f(m,x)=\{x\}$ and $R\cap\varphi_f(m,x)\neq\emptyset$, we deduce that $x\in R$. So, $R\cap U\neq\emptyset$. Finally, observe that $R\subset\bigcup\mathcal{V}_f(m,S)\subset\bigcup\mathcal{U}$. Therefore, $R\in\langle\mathcal{U}\rangle_c$ and this finishes the proof. □

**Theorem 3.36.** *Let $X$ be a sequential space such that $L_X$ is discrete cellular in $X$. If $X$ is a $\gamma$-space, then $\mathcal{S}_c(X)$ is a $\gamma$-space.*

*Proof.* Let $\lceil f, g \rceil$ a $\gamma$-structure for $X$. We will verify that $\lceil \psi_f, \psi_g \rceil$ is a $\gamma$-structure for $\mathcal{S}_c(X)$. By (3) of Lemma 3.31, $\psi_f$ and $\psi_g$ are well defined. One can show that $\psi_f$ and $\psi_g$ have $(2)_{\psi_f}$ and $(2)_{\psi_g}$ using similar arguments to those exhibited in the proof of Theorem 3.35. According to (7) of Lemma 3.31 the functions $\psi_f$ and $\psi_g$ have $(4)_{\psi_f}$ and $(4)_{\psi_g}$. To finish the proof, we take $(m, S) \in \mathbb{N} \times \mathcal{S}_c(X)$ and $Q \in \mathcal{S}_c(X)$ such that $Q \in \psi_g(m, S)$ and we will see that $\psi_g(m, Q) \subset \psi_f(m, S)$. We start by taking $P \in \psi_g(m, Q)$. Let $y \in (S \cap L_X) \cup D_f(m, S)$. To prove that $P \cap \varphi_f(m, y) \neq \emptyset$, we take three cases.

**Case I.** $y \in Q \cap L_X$.

We observe that $\varphi_g(m, y) \subset \varphi_f(m, y)$. Since $P \cap \varphi_g(m, y) \neq \emptyset$, we deduce that $P \cap \varphi_f(m, y) \neq \emptyset$.

**Case II.** $y \in (S \cap L_X) - (Q \cap L_X)$.

The fact that $Q \in \psi_g(m, S)$ guarantees that we can take $q \in Q \cap \varphi_g(m, y)$. We observe that $q \in \varphi_g(m, q) \cap \varphi_g(m, y)$. In light of (1) of Lemma 3.31, either $q \notin L_X$ or $q = y$. If $q$ and $y$ were equal, then $y$ would be an element of $Q \cap L_X$, an absurd. So, we have $q \notin L_X$. Then, $q \in D_g(m, Q)$. Since $P \cap \varphi_g(m, q) \neq \emptyset$ and $\varphi_g(m, q) = \{q\}$, we deduce that $q \in P$. Also, $q \in \varphi_g(m, y) \subset \varphi_f(m, y)$. Hence, $P \cap \varphi_f(m, y) \neq \emptyset$.

**Case III.** $y \in D_f(m, S)$.

First, we notice that the fact that $\varphi_f(m, y) = \{y\}$ implies that $\varphi_g(m, y) = \{y\}$. So, $y \in D_g(m, S)$. Since $Q \cap \varphi_g(m, y) \neq \emptyset$, $y \in Q$. From (4) of Lemma 3.31, we obtain that $D_g(m, S) \subset S - \bigcup_{w \in Q \cap L_X} \varphi_g(m, w)$. Thus, $y \in D_g(m, Q)$. Using that $P \cap \varphi_g(m, y) \neq \emptyset$ and the fact that $\varphi_g(m, y) = \{y\}$, we conclude that $y \in P$. This means $P \cap \varphi_f(m, y) \neq \emptyset$.

We observe that $D_g(m, S) \subset \bigcup_{x \in S \cap L_X} \varphi_f(m, x) \cup D_f(m, S)$ and $\varphi_g(m, y) \subset \varphi_f(m, y)$ for all $y \in S \cap L_X$. We infer that $\bigcup \mathcal{V}_g(m, S) \subset \bigcup \mathcal{V}_f(m, S)$. Now, according to (5) of Lemma 3.31, $\bigcup \mathcal{V}_g(m, Q) \subset \bigcup \mathcal{V}_g(m, S)$. Thus, $P \subset \bigcup \mathcal{V}_g(m, Q) \subset \bigcup \mathcal{V}_f(m, S)$. Apply Lemma 3.30 to get that $P \in \psi_f(m, S)$. We conclude that $\psi_g(m, Q) \subset \psi_f(m, S)$. □

**Question 3.37.** Can the discrete cellularity on $L_X$ be omitted in theorems 3.35 and 3.36?

For a space $X$, define $N_X$ as the set of all elements $x \in L_X$ such that there exists an open closed subset $V$ of $X$ satisfying $x \in L_X$ and $L_X - V \neq \emptyset$.

**Lemma 3.38.** *Let $P$ be an additive and hereditary with respect to closed subsets topological property. If $X$ is a space such that $|L_X| \geq 2$ and $N_X \neq \emptyset$, then the condition $\mathcal{S}_c(X)$ has $P$ implies $X$ has $P$.*

*Proof.* Let $x \in N_X$ and let $V$ be an open closed subset of $X$ such that $x \in L_X$ and $L_X - V \neq \emptyset$. In light of [5, Proposition 2.2.4, p. 74], the space $X$ is equal to the topological sum $V \oplus (X - V)$. On the other hand, we may choose $S, Q \in \mathcal{S}_c(X)$ in such a way $\lim S = x$, $S \subset V$ and $Q \cap V = \emptyset$. Now, according to Lemma 2.2, $E(S, V)$ and $E(Q, X - V)$ are closed subsets of $\mathcal{S}_c(X)$ homeomorphic to $V$ and $X - V$, respectively. Since $P$ is hereditary with respect to closed subsets, we have that $V$ and $X-V$ have $P$. Finally, the fact that $X$ has $P$ is a consequence of the assumption that $P$ is additive. □

**Theorem 3.39.** *Let $X$ be a space such that $|L_X| \geq 2$ and $N_X \neq \emptyset$. Each one of the following statements holds.*

(1) *If $\mathcal{S}_c(X)$ is developable, then $X$ is developable.*
(2) *If $\mathcal{S}_c(X)$ has a $G_\delta$-diagonal, then $X$ has a $G_\delta$-diagonal.*
(3) *If $\mathcal{S}_c(X)$ is Moore, then $X$ is Moore.*
(4) *If $\mathcal{S}_c(X)$ is an $\alpha$-space, then $X$ is an $\alpha$-space.*
(5) *If $\mathcal{S}_c(X)$ is a strongly first countable space, then $X$ is a strongly first countable space.*
(6) *If $\mathcal{S}_c(X)$ is a $\gamma$-space, then $X$ is a $\gamma$-space.*
(7) *If $\mathcal{S}_c(X)$ is Lãsnev, then $X$ is Lãsnev.*
(8) *If $\mathcal{S}_c(X)$ is paracompact, then $X$ is paracompact.*

*Proof.* The result follows from Lemma 3.38 and the fact that each one of the following properties is additive and hereditary with respect to closed subsets: being developable space, having a $G_\delta$-diagonal, being Moore space, being $\alpha$-space, being strongly first countable space, being $\sigma$-space, being $\gamma$-space, being Lãsnev space and being paracompact. □

**Question 3.40.** Can the assumptions on $X$ be removed in Theorem 3.39?

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# Conclusiones

El hiperespacio de todas las sucesiones convergentes no triviales de un espacio topológico Hausdorff $X$ es denotado por $\mathcal{S}_c(X)$. Dado que el estudio de los hiperespacios resulta sumamente útil para determinar las propiedades de los espacios en lo que éstos se basan y viceversa nuestro proyecto de investigación está enfocado en estudiar la preservación y la reversibilidad en el hiperespacio de sucesiones convergentes no triviales de una serie de propiedades topológicas que generalizan a los espacios métricos. Específicamente estudiamos las siguientes propiedades: ser espacio Lãsnev, ser $\aleph_0$-espacio, ser $\alpha$-espacio, ser espacio cósmico, ser espacio desarrollable, ser espacio Moore, ser $\gamma$-espacio, ser $M_2$-espacio, ser $M_3$-espacio, ser espacio Nágata, ser $\sigma$-espacio, ser espacio estrictamente primero numerable, ser espacio hereditariamente normal, ser espacio perfectamente normal, ser espacio normal, ser espacio regular y ser espacio paracompacto.

Concluimos que con el estudio antes mencionado hemos logrado ampliar el conocimiento acerca del hiperespacio de sucesiones convergentes no triviales, lo cual ayudará al entendimiento de este novedoso hiperespacio y contribuirá a hacer más completo el entendimiento sobre las relaciones entre los hiperespacios y los espacios base.

Específicamente, logramos probar que si un espacio topológico $X$ tiene alguna de las siguientes propiedades: ser espacio desarrollable, ser espacio Moore, ser $\aleph_0$-espacio, entonces su hiperespacio $\mathcal{S}_c(X)$ tambien la posee y que si $\mathcal{S}_c(X)$ es espacio cósmico, entonces $X$ es espacio cósmico. Adicional a esto, mostramos espacios $X$ que prueban que las propiedades ser espacio Lãsnev, ser espacio paracompacto, ser espacio hereditariamente normal, ser espacio perfectamente normal y tener una $G_\delta$-diagonal no satisfacen la siguiente condición: si $X$ tiene la propiedad, entonces el hiperespacio $\mathcal{S}_c(X)$ tiene la propiedad. Logramos hallar las condiciones necesarias o suficientes para establecer una relacion entre el hecho que $\mathcal{S}_c(X)$ es $\aleph_0$-espacio, $\alpha$-espacio, espacio perfectamente normal, espacio normal, espacio regular, espacio hereditariamente normal, $M_2$-espacio, $M_3$-espacio, $\sigma$-espacio, espacio Nágata, espacio desarrollable, espacio Moore, espacio estrictamente primero numerable, espacio Lãsnev, $\gamma$-espacio o espacio paracompacto y el hecho que el espacio $X$ es $\aleph_0$-espacio, $\alpha$-espacio, espacio perfectamente normal, espacio normal, espacio regular, espacio hereditariamente normal, $M_2$-espacio, $M_3$-espacio, $\sigma$-espacio, espacio Nágata, espacio desarrollable, espacio Moore, espacio estrictamente primero numerable, espacio Lãsnev, $\gamma$-espacio o espacio paracompacto. Finalmente, expusimos bajo que condiciones del espacio $X$ es posible probar que las propiedades $\alpha$-espacio, $\gamma$-espacio y espacio estrictamente primero numerable satisfacen la condición: si $X$ tiene la propiedad, enotonces el hiperespacio $\mathcal{S}_c(X)$ tiene la propiedad.

Es importante mencionar que al ser $\mathcal{S}_c(X)$ un hiperespacio tan novedoso aún queda bastante campo de estudio y resulta de sumo interés descubrir las relaciones que mantiene con otros hiperespacios.

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