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R^i-SETS, PSEUDO–CONTRACTIBILITY AND WEAK CONTRACTIBILITY ON HYPERSPACES OF CONTINUA

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Abstract. In this paper we discuss the notions of pseudo-contractibility and weak contractibility on hyperspaces of (Hausdorff) continua. Also we prove that if a continuum X contains an R^i-set then it is not pseudo-contractible. As a consequence we have that the existence of an R^i-set in a continuum X implies non(pseudo)-contractibility of some hyperspaces.

1. Introduction

R. H. Bing introduced the notion of pseudo-contractibility. However, W. Kuperberg gave the first example which proves that the notions of pseudo-contractibility and contractibility are different. This example was never published by himself but it is known among continuum theorists. He asked whether or not the sin $\frac{1}{x}$–curve is pseudo-contractible. H. Katsuura proves in [11] that the sin$\frac{1}{x}$–curve is not pseudo-contractible with factor space itself. In the same paper he proves that if the factor space Y is a non-degenerate indecomposable continuum such that each one of its composants is arc-wise connected, and if X is a continuum having a proper non-degenerate arc component, then X is not pseudo-contractible with factor space Y.

W. Dębski shows in [7] that the sin $\frac{1}{x}$–curve is not pseudo-contractible. On the other hand, M. Sobolewsky proves that the only chainable continuum that is pseudo-contractible is the arc, see [19]. In particular the pseudo-arc is another example of a non pseudo-contractible continuum. In [3] there is a general study about pseudo-homotopies and pseudo-contractibility. The interested reader is referred to [2, 3, 7, 9, 11] and [19].

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In [18] D. G. Paulowich studies the concepts of contractibility and weak contractibility on Hausdorff continua and their hyperspaces. He used the same ideas used in metric continua to define contractibility and weak contractibility in Hausdorff continua (the factors spaces in those concepts are Hausdorff continua) as well as we have defined here contractibility and pseudo-contractibility with metric continua. These concepts are different, however both concepts are equivalent when we use metric continua.

The present paper is divided in five sections. After the introduction and preliminaries, in section 3 we present results about weak contractibility and pseudo-contractibilty on hyperspaces, mainly we show that the concepts of weak contractibility (pseudo-contractibility) and contractibility are equivalent in some hyperspaces. In section 4 we prove that if a continuum X contains an R^i-set then it is not pseudo-contractible. As a consequence we have that the existence of an R^i-set in a continuum X implies non(pseudo)-contractibility of some hyperspaces. In section 5 we present some questions regarding pseudo-contractibility on hyperspaces.

2. Preliminaries

A *continuum* means a nonempty compact connected metric space. A nonempty compact connected Hausdorff space is called *Hausdorff continuum.* A *map* is a continuous function. If there exists a homeomorphism $f : X \to Y$, we say that X is homeomorphic to Y and we write $X \approx Y$. A continuum X is said to be *unicoherent* provided that for each pair of subcontinua H and K of X such that $X = H \cup K$, $H \cap K$ is connected, and it is *hereditarily unicoherent* if each subcontinuum of X is unicoherent. An *arc* is understood as a homeomorphic image of a closed unit interval $I = [0, 1]$. A space Z is said to be *arcwise-connected* provided each pair of points of a space Z can be joined by an arc lying in Z. A *curve* is a one-dimensional continuum.

A continuum X is *arc-like (circle-like)* provided that for each $\varepsilon > 0$, there exists an ε-map $f : X \to [0, 1]$ ($f : X \to S^1$, where S^1 is the unit circle). A *proper circle-like continuum* is a circle-like continuum which is not an arc-like continuum. If $A \subset X$, here $Bd(A)$ denotes the boundary of A in X, and $\overline{A}$ denotes the closure of A in X.

Let X be a Hausdorff continuum. The hyperspace of all nonempty closed subsets of X is denoted by 2^X, the hyperspace of all subcontinua of X is denoted by $C(X)$. If $n \in \mathbb{N}$, the hyperspace of all nonempty closed subsets of X with at most n components is denoted by $C_n(X)$, the hyperspace of all nonempty subsets of X with at most n points is denoted by $F_n(X)$, in particular $F_1(X)$ is called the hyperspace of singletons of X and it is homeomorphic to X, $F_\infty(X)$ denotes the hyperspace of all finite subsets of X and $C_\infty(X)$ denotes the hyperspace of all closed subsets of X with a finite number of components. Note that $C_1(X) = C(X)$, $F_1(X) \subset C(X) \subset C_n(X) \subset C_\infty(X) \subset 2^X$,

$F_1(X) \subset F_n(X) \subset F_\infty(X) \subset C_\infty(X) \subset 2^X$ and $F_n(X) \subset C_n(X)$. If X is a Hausdorff continuum these hyperspaces are endowed with the *Vietoris Topology*. If X is a metric continuum these hyperspaces are also endowed with the *Hausdorff metric* defined by: $\mathcal{H}_d(A,B) = \inf\{\varepsilon > 0 | A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(A)\}$, where $N_\varepsilon(Y) = \{x \in X|$ there exists $y \in Y$ such that $d(y,x) < \varepsilon\}$, $\varepsilon > 0$ and $Y \in 2^X$. It is well known that $\mathcal{H}_d(A,B) < \varepsilon$ if and only if $A \subset N_\varepsilon(B)$ and $B \subset N_\varepsilon(A)$.

The following are the usual definitions related with contractibility (see for example [20, p. 225], [13, pp. 370, 374], [10, pp. 155, 156]).

Definition 2.1. *Let X and Y be topological spaces and let $f,\ g : X \to Y$ be maps. We say that f is homotopic to g (or f and g are homotopic), written $f \simeq g$, if there exists a map $H : X \times I \to Y$, called homotopy, fulfilling $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$.*

Definition 2.2. *A topological space X is said to be:*

a) *contractible if its identity map is homotopic to a constant map in X;*
b) *contractible with respect to a topological space Y if each map $f : X \to Y$ is homotopic to a constant map.*

A subspace Z of X is said to be contractible in X if the inclusion map from Z into X is homotopic to a constant map in X.

It is not difficult to prove that X is contractible if and only if X is contractible with respect to Y, for every space Y.

Following the classical definitions of contractibility, concepts related with pseudo-contractibility are defined as follows.

Definition 2.3. *Let X and Y be topological spaces and let $f,\ g : X \to Y$ be maps. We say that f is pseudo-homotopic to g (or f and g are pseudo-homotopic) if there exist a continuum C, points $a,\ b \in C$ and a map $H : X \times C \to Y$ fulfilling $H(x,a) = f(x)$ and $H(x,b) = g(x)$ for each $x \in X$. The continuum C is called a factor space. The map H is called a pseudo-homotopy between f and g. We write $f \simeq_C g$ to say that f is pseudo-homotopic to g, where C denotes a factor space.*

Definition 2.4. *A topological space X is said to be:*

a) *Pseudo-contractible if its identity map is pseudo-homotopic to a constant map in X,*
b) *pseudo-contractible with respect to a topological space Y if each map $f : X \to Y$ is pseudo-homotopic to a constant map.*

A subspace Z of X is said to be pseudo-contractible in X if the inclusion map from Z into X is pseudo-homotopic to a constant map in X.

In [18] D. G. Paulowich studied the contractibility and weak contractibility using the next definition.

Definition 2.5. *Let X and Y be topological spaces and let $f,\ g : X \to Y$ be maps. We say that f is homotopic (weakly homotopic) to g, or f and g are homotopic (weakly homotopic), if there exist a Hausdorff arc (Hausdorff continuum) K, points $a,\ b \in K$ and a map $H : X \times K \to Y$ fulfilling $H(x,a) = f(x)$ and $H(x,b) = g(x)$ for each $x \in X$. The Hausdorff continuum K is called a factor space. The map H is called a homotopy (weak homotopy) between f and g.*

In a similar way as in Definition 2.4, the following is defined:

Definition 2.6. *A topological space X is said to be:*

a) *(weakly) contractible if its identity map is (weakly) homotopic to a constant map in X;*
b) *(weakly) contractible with respect to a topological space Y if each map $f : X \to Y$ is (weakly) homotopic to a constant map.*

A subspace Z of X is said to be (weakly) contractible in X if the inclusion map from Z into X is (weakly) homotopic to a constant map in X.

It is clear that in each case contractibility implies pseudo-contractibility and pseudo-contractibility implies weakly contractibility.

Remark 2.7. Let $W \subset Z \subset X$. If Z is weakly contractible in X, then W is weakly contractible in X. If Z is weakly contractible in X and $X \subset X'$ then Z is weakly contractible in X'. If X is weakly contractible, then every subspace Z of X is weakly contractible in X.

3. Hyperspaces, weak contractibility and pseudo-contractibility

In [18, Theorems 3 and 4 and Corollary] D. G. Paulowich proves that if Y is a compact subspace of X and X is a compact space such that $F_1(X)$ is a retract of $C(X)$, then Y weakly contractible in X implies Y is contractible in X. He also proves that if X is a Hausdorff continuum, the following three statements are equivalent:

1. $F_1(X)$ is contractible in 2^X.
2. 2^X is contractible using an order preserving homotopy.
3. $C(X)$ is contractible using an order preserving homotopy.

As a corollary he proves that if X is a weakly contractible Hausdorff continuum, then 2^X and $C(X)$ are contractible using an order preserving homotopy, ([18, Corollary, p. 44]).

Therefore we have that, if X is a contractible metric continuum, then the hyperspaces $C(X)$ and 2^X are contractible (see also [17, Corollary 16.8]).

Additionally we get as a consequence of [18, Theorem 3] the following result.

Proposition 3.1. *If X is a compact Hausdorff space such that $F_1(X)$ is a retract of $C(X)$ and X is weakly contractible, then X is contractible.*

The last proposition gives a partial answer to [3, Question 64].

The main problem in this section is to prove that the notions of weak contractibility (pseudo-contractibility) and contractibility coincide in the hyperspaces 2^X, $C(X)$, $C_\infty(X)$ and $C_n(X)$ for any $n \in \mathbb{N}$.

Theorem 3.2. *Let X be a Hausdorff continuum. If 2^X is weakly contractible then it is contractible.*

Proof. Since 2^X is weakly contractible, then by [18, Corollary, p. 44], 2^{2^X} is contractible and since $F_1(2^X)$ is a retract of 2^{2^X} ([12, Lemma 1.1]) we have by [3, Theorem 19] that $F_1(2^X)$ is contractible. Since $F_1(2^X) \approx 2^X$, 2^X is contractible. $\square$

Let Z be a compact Hausdorff space. Using the fact that $F_1(2^Z)$ is a retract of $C(2^Z)$, we also prove the following result taking Y as 2^Z and X as $F_1(Z)$ in [18, Theorem 3].

Theorem 3.3. *If $F_1(Z)$ is weakly contractible in 2^Z, then $F_1(Z)$ is contractible in 2^Z.*

Corollary 3.4. *Let X be a continuum. If $F_1(X)$ is weakly contractible in 2^X, then 2^X contractible.*

The following corollary is a consequence of [18, Theorem 4], [8, Exercise 9.7], Theorem 3.2, Theorem 3.3 and Corollary 3.4.

Corollary 3.5. *Let X be a Hausdorff continuum, the following sentences are equivalent:*

1. *$F_1(X)$ is contractible in $C(X)$.*
2. *$F_1(X)$ is contractible in 2^X.*
3. *2^X is contractible (using an order preserving homotopy).*
4. *$C(X)$ is contractible (using an order preserving homotopy).*
5. *$F_1(X)$ is contractible in $C_n(X)$ for some $n \in \mathbb{N}$.*
6. *$C_n(X)$ is contractible for each $n \in \mathbb{N}$.*
7. *$C_n(X)$ is contractible for some $n \in \mathbb{N}$.*
8. *$F_1(X)$ is contractible in $C_\infty(X)$.*
9. *$C_\infty(X)$ is contractible.*
10. *$F_1(X)$ is weakly contractible in 2^X.*
11. *$F_1(X)$ is weakly contractible in $C(X)$.*
12. *$C(X)$ is weakly contractible.*
13. *2^X is weakly contractible.*
14. *$F_1(X)$ is weakly contractible in $C_n(X)$ for some $n \in \mathbb{N}$*
15. *$C_n(X)$ is weakly contractible for each $n \in \mathbb{N}$.*
16. *$C_n(X)$ is weakly contractible for some $n \in \mathbb{N}$.*

17. *$F_1(X)$ is weakly contractible in $C_\infty(X)$.*
18. *$C_\infty(X)$ is weakly contractible.*

COROLLARY 3.6. *Let X be a Hausdorff continuum. If X is weakly contractible then we have that statements 1-18 of Corollary 3.5 hold.*

COROLLARY 3.7. *Let X be a continuum. If X is pseudo- contractible then we have that statements 1-18 of Corollary 3.5 hold.*

The converse of Corollary 3.7 is not true. We will consider two examples, the first one is an unicoherent continuum and the other one is not a unicoherent continuum. We consider the space $X = X_0 \cup S^1 \subset \mathbb{C}$, where $X_0 = \{\frac{t+2}{t+1}e^{it} : t \in [0,\infty)\}$ is the spiral approaching to the unit circle S^1 (the symbol $\mathbb{C}$ denotes the set of the complex numbers). We know that $C(X)$ is contractible, because $C(X)$ is homeomorphic to the $cone(X)$. However X is not pseudo-contractible, because, by [3, Theorem 59] any pseudo-contractible curve is hereditarily unicoherent. On the other hand S^1 is a continuum such that $C(S^1)$ is contractible. Nevertheless, by [3, Corollary 56], S^1 is not pseudo-contractible.

Now we will discuss the weak contractibility and pseudo-contractibility in the hyperspaces $F_n(X)$ and $F_\infty(X)$.

THEOREM 3.8. *Let X be a continuum. The space $F_1(X)$ is pseudo-contractible in $F_\infty(X)$ if and only if $F_\infty(X)$ is pseudo-contractible.*

PROOF. Assume that there exist a continuum C, points $a,\ b \in C$ and a map $H : F_1(X)\times C \to F_\infty(X)$, such that $H(\{k\}, a) = \{k\}$ and $H(\{k\}, b) = A_0$ for some $A_0 \in F_\infty(X)$ and each $\{k\} \in F_1(X)$.

Consider the function $G : F_\infty(X) \times C \to F_\infty(X)$ defined by $G(K, c) = \bigcup\{H(\{k\}, c) : k \in K\}$. Notice that $G(K, c) \in F_\infty(X)$ because K is a finite subset of X. Since H is a map, thus by [17, 1.48], G is well defined and we have that:

$$G(K, a) = \bigcup\{H(\{k\}, a) : k \in K\} = \bigcup\{\{k\} : k \in K\} = K,$$

$$G(K, b) = \bigcup\{H(\{k\}, b) : k \in K\} = A_0 \text{ for each } K \in F_\infty(X).$$

Let $A,\ B \in F_\infty(X)$ and let $s,\ t \in C$ such that $\mathcal{H}_d(A, B) < \delta$ and $d(s, t) < \delta$. We will show that $\mathcal{H}_d(G(A, t), G(B, s)) < \varepsilon$, by proving that $G(A, t) \subset N_\varepsilon(G(B, s))$ and $G(B, s) \subset N_\varepsilon(G(A, t))$. Let $p' \in G(A, t)$, then there exists a point $p \in A$ such that $p' \in H(\{p\}, t)$. Since $A \subset N_\delta(B)$ there exists $q \in B$ such that $d(p, q) < \delta$. So, $\mathcal{H}_d(\{p\}, \{q\}) < \delta$. On the other hand since $d(t, s) < \delta$ we have that $\mathcal{H}_d(H(\{p\}, t), H(\{q\}, s)) < \varepsilon$. Therefore, $p' \in H(\{p\}, t) \subset N_\varepsilon(H(\{q\}, s)) \subset N_\varepsilon(G(B, s))$. Then $p' \in N_\varepsilon(G(B, s))$. Hence $G(A, t) \subset N_\varepsilon(G(B, s))$. Analogously $G(B, s) \subset N_\varepsilon(G(A, t))$.

The other implication is trivial. ∎

Corollary 3.9. *Let X be a continuum. If X is a pseudo-contractible continuum then $F_\infty(X)$ is pseudo-contractible.*

The converse of this result is not true. By [8, Exercise 9.8], $F_\infty(S^1)$ is contractible but S^1 is not pseudo-contractible.

Corollary 3.10. *Let X be a continuum. If $F_\infty(X)$ is pseudo-contractible then we have that statements 1-18 of Corollary 3.5 hold.*

Proof. Since $F_\infty(X)$ is pseudo-contractible, $F_1(X)$ is pseudo-contractible in $F_\infty(X)$ and therefore, $F_1(X)$ is pseudo-contractible in 2^X. □

Using the fact, $F_n(X) \subset F_\infty(X)$, we get the following.

Corollary 3.11. *Let X be a continuum. If $F_1(X)$ is pseudo-contractible in $F_n(X)$ for some $n \in \mathbb{N}$, then we have that statements 1-18 of Corollary 3.5 hold.*

Theorem 3.12. *Let X be a Hausdorff continuum. If X is weakly contractible, then $F_n(X)$ is weakly contractible for any $n \in \mathbb{N}$.*

Proof. Since X is weakly contractible, there exist a Hausdorff continuum C, points $a, b \in C$ and a map $H : X \times C \to X$ such that $H(x, a) = x$ and $H(x, b) = x_0$ for each $x \in X$.

Let $n \in \mathbb{N}$, the function $G : F_n(X) \times C \to F_n(X)$ defined by $G(\{x_1, \ldots, x_n\}, c) = \{H(x_1, c), \ldots, H(x_n, c)\}$ is a weak homotopy between the identity in $F_n(X)$ and the constant map $\{x_0\}$, i.e.,

$$G(\{x_1, \ldots, x_n\}, a) = \{H(x_1, a), \ldots, H(x_n, a)\} = \{x_1, \ldots, x_n\}$$

and $G(\{x_1, \ldots, x_n\}, b) = \{H(x_1, b), \ldots, H(x_n, b)\} = \{x_0\}$. □

Corollary 3.13. *Let X be a continuum. If $F_n(X)$ is weak contractible for some $n \in \mathbb{N}$ then we have that statements 1-18 of Corollary 3.5 hold.*

Corollary 3.14. *Let X be a continuum. If $F_n(X)$ is pseudo-contractible for some $n \in \mathbb{N}$ then we have that statements 1-18 of Corollary 3.5 hold and $F_\infty(X)$ is pseudo-contractible too.*

Proof. Since $F_n(X)$ is pseudo-contractible, thus $F_1(X)$ is pseudo-contractible in $F_n(X)$ and $F_1(X)$ is pseudo-contractible in $F_\infty(X)$. □

The converse of Corollary 3.14 is not true. We give two examples; one of them is not a unicoherent continuum with $F_n(X)$ non-pseudo-contractible for every $n > 1$ and the other one is a unicoherent continuum with $F_2(X)$ non-pseudo-contractible. To see the first example, we consider $X = S^1$; it is known by [8, Exercise 9.8] that $F_\infty(S^1)$ is contractible, and 1-18 hold because $C(S^1)$ is contractible. On the other hand, we know that if $n > 1$, $F_{2n+1}(S^1)$ is homotopically equivalent to S^{2n+1} (see [6, Theorem 4.1]) and $F_{2n}(S^1)$ is homotopically equivalent to S^{2n-1} (see [6, Theorem 4.2]). Since S^n is ANR for all $n \in \mathbb{N}$ and S^n is not contractible, then by [3, Corollary 46], S^n is not pseudo-contractible. Therefore, by [3, Corollary 39], $F_n(S^1)$ is not pseudo-contractible for all $n \in \mathbb{N}$.

For the second example we consider $X = S_1 \cup Y \cup S_2 \subset \mathbb{C}$, where

$$Y = \left\{ \left(\frac{t}{1+|t|} + 2\right)e^{it} : t \in \mathbf{R} \right\},\ S_1 = \{e^{it} : t \in \mathbb{R}\} \text{ and } S_2 = \{3e^{it} : t \in \mathbf{R}\}.$$

X is the union of two circles and a spiral which surrounds them asymptotically. $C(X)$ is contractible, because X is a Kelley's continuum (see [8, Theorem 9.4, p. 129]). By [4], $F_2(X)$ is not unicoherent and by [3, Corollary 56], it is not pseudo-contractible.

4. R^i-Sets and pseudo-contractibility

It is well known by [1, Corollary 3.3] if a continuum X has an R^i-set $i = 1, 2, 3$, then X is not contractible. The main result of this section is to prove that if a continuum X has an R^i-set, then X is not pseudo-contractible. Let us recall some definitions.

Definition 4.1. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of a space X.*

$$\liminf A_n = \{x \in X : \textit{for each open } U \subset X \textit{ such that } x \in U,\ U \cap A_n \neq \emptyset \textit{ for all but finitely many } n\}.$$

$$\limsup A_n = \{x \in X : \textit{for each open } U \subset X \textit{ such that } x \in U,\ U \cap A_n \neq \emptyset \textit{ for infinitely many } n\}.$$

Let $A \in 2^X$, we write $\lim A_n = A$ to mean $\liminf A_n = A = \limsup A_n$.

Definition 4.2. *A nonempty closed proper subset K of a continuum X is called*

- *R^1-set if there exist an open set U containing K and two sequences $\{C_n^i\}_{n\in\mathbb{N}}$, $i = 1, 2$ of components of U such that $K = \limsup C_n^1 \cap \limsup C_n^2$,*
- *R^2-set if there exist an open set U containing K and two sequences $\{C_n^i\}_{n\in\mathbb{N}}$, $i = 1, 2$ of components of U such that $K = \lim C_n^1 \cap \lim C_n^2$,*
- *R^3-set if there exist an open set U and a sequence $\{C_n\}_{n\in\mathbb{N}}$ of components of U such that $K = \liminf C_n$.*

The following theorem is due to W. J. Charatonik and discussed with us in México, where we reviewed some details to the final version.

Theorem 4.3. *Let X be a continuum. If X contains an R^i-set, $i = 1, 2, 3$ then X is not pseudo-contractible.*

Proof. Since each R^2-set is an R^3-set (see [1, Theorem 2.3]) and each R^1-set contains an R^3-set (see [1, Theorem 2.5]), it is enough to prove the theorem when X contains an R^3-set. Let A be an R^3-set of X. By definition, there exist an open set U and a sequence $\{C_n\}_{n\in\mathbb{N}}$ of components of U such that $A = \liminf C_n$. Since $A \subset U$ there exists an $\varepsilon > 0$ such that $d(A, X \setminus U) > \varepsilon$.

Suppose that $H : X \times Y \to X$ is a pseudo-homotopy such that there exist a continuum Y and points $a, b \in Y$ such that $H(x, a) = x$ and $H(x, b) = x_0$ for all $x \in X$ and $x_0 \in X$. Since H is uniformly continuous, there exists $\delta > 0$ such that if $\operatorname{diam}(K) < \delta$ then $\operatorname{diam}(H(K \times \{t\})) < \varepsilon$, for each $t \in Y$.

Now consider the set $P = \{c, c_1, \dots, c_n, \dots\} \subset U$, where $c \in A$, $c_i \in C_i$ for each $i > 0$ and $\lim c_i = c$. Without loss of generality we can assume that $\operatorname{diam}(P) < \delta$. Let $V = \{t \in Y : H(P \times \{t\}) \subset U\}$. The set V satisfies the following conditions:

1. $V \neq \emptyset$;
2. $V \neq Y$;
3. V is an open set of Y.

The set $V \neq \emptyset$ because $a \in V$. $V \neq Y$ because if $V = Y$ then $b \in V$ and $H(P \times \{b\}) = \{x_0\} \subset U$. Let C be the component of U containing x_0. Then we consider a component C_j of U such that $C_j \neq C$. Since $c_j \in C_j$, $H(\{c_j\} \times Y) \subset U$ and $H(\{c_j\} \times Y)$ is a connected set containing x_0 and c_j, a contradiction. Hence, $V \neq Y$. Finally, by continuity of H, V is an open set of Y.

Let V_0 be the component of V containing a. Since $H(\{c_i\} \times V_0)$ is a connected set containing c_i, then $H(\{c_i\} \times V_0) \subset C_i$ for all $i = 1, 2, \dots$ In other words, $H((c_i, t) \in C_i$ for all $i = 1, 2, \dots$ and all $t \in V_0$.

Since $\lim H(c_i, t) = H(c, t)$, then $H(c, t) \in \liminf C_i = A$ for all $t \in V_0$.

On the other hand, if $t_0 \in Bd(V_0)$ there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset V_0$ such that $\lim t_n = t_0$. So, $\lim H(c, t_n) = H(c, t_0) \in A$, because A is a closed set.

Finally, since V_0 is a component of the open set V, by [16, 5.7 Boundary Bumping Theorem III, p. 75], $\overline{V_0} \cap (Y \setminus V) \neq \emptyset$. Let $t' \in \overline{V_0} \cap (Y \setminus V)$, then $t' \in Bd(V_0) \setminus V$, i.e., $H(P \times \{t'\}) \not\subset U$; in this way, there exists $d \in P$ such that $H(d, t') \notin U$, but $H(c, t') \in A$. Therefore, $d(H(c, t'), H(d, t')) > \varepsilon$, a contradiction, because c, $d \in P$ and $\operatorname{diam}(H(P \times \{t'\})) < \varepsilon$. □

From Corollaries 3.5, 3.10, 3.14 and Theorem 4.3 we have the following result.

THEOREM 4.4. *Let X be a continuum. If X has an R^3-set, then*

1. *2^X is not (pseudo-)contractible,*
2. *$C(X)$ is not (pseudo-)contractible,*
3. *$C_n(X)$ is not (pseudo-)contractible for each $n \in \mathbb{N}$,*
4. *$C_\infty(X)$ is not (pseudo-)contractible,*
5. *$F_n(X)$ is not pseudo-contractible for each $n \in \mathbb{N}$,*
6. *$F_\infty(X)$ is not pseudo-contractible.*

Finally we present some continua X such that its $F_2(X)$ is not pseudo-contractible. First of all we recall the following definition.

DEFINITION 4.5. *Let X and Y be topological spaces. A map $f : X \to Y$ is said to be essential provided that f is not homotopic to any constant map of X into Y. A map $f : X \to Y$ is said to be inessential provided that f is not essential.*

It is well known by [15, Theorem 3.4] that if X is a proper circle-like continuum, then $F_2(X)$ does not have trivial shape. Therefore, if X is a proper circle-like continuum, then by [3, Theorem 48], $F_2(X)$ is not pseudo-contractible. On the other hand, by [15, Theorem 3.3] if there exists an essential map from a continuum X onto S^1, then there exists an essential map from $F_2(X)$ onto S^1, in other words, $F_2(X)$ does not have trivial shape and therefore, by [3, Theorem 52], $F_2(X)$ is not pseudo-contractible. In particular $F_2(X)$ is not pseudo-contractible if X is a decomposable homogeneous curve or hereditarily decomposable homogeneous continuum, because by [14, Lemma 5.1] and [14, Lemma 5.2] there exists an essential map from X onto S^1.

5. QUESTIONS

In this section we ask some questions regarding this topic.

It is pertinent to recall the following question (see [10, Question 78.20, p. 402]).

QUESTION 5.1. *Let X be a continuum. If $F_n(X)$ is contractible for some $n \in \mathbb{N}$, is then X contractible?*

In general we have the following.

QUESTION 5.2. *Let X be a continuum. If $F_n(X)$ is weakly contractible (pseudo-contractible) for some $n \in \mathbb{N}$, is then X weakly contractible (pseudo-contractible)?*

In Section 3, we have showed that for any Hausdorff continuum (metric continuum) X, the concepts of weakly contractibility (pseudo-contactibility) and contractibility are the same in the hyperspaces 2^X, $C(X)$, $C_n(X)$ and $C_\infty(X)$.

Let $H(X) \in \{F_\infty(X),\ F_n(X),\ C_\infty(X),\ C_n(X),\ 2^X\}$, $n \in \mathbb{N}$. The following question appears in [5, Question 3.11, p. 755].

QUESTION 5.3. *Let X be a continuum. What are necessary and /or sufficient conditions in terms of X in order that $H(X)$ is or is not contractible?*

Partial answer is given in Corollary 3.6.

QUESTION 5.4. *(In the sense of [18]) Let X be a Hausdorff continuum. What are necessary and /or sufficient conditions in terms of X in order that $H(X)$ is or is not contractible?*

Partial answers are given in [18, Theorems 5 and 6].

Now for the following two questions we consider

$$H(X) \in \{F_\infty(X),\ F_n(X)\}, \qquad n \in \mathbb{N}.$$

QUESTION 5.5. *Let X be a continuum. What are necessary and /or sufficient conditions in terms of X in order that $H(X)$ is or is not weakly contractible (pseudo-contractible)?*

Partial answers are given in Corollary 3.9 and Theorem 3.12.

QUESTION 5.6. *What kinds of Hausdorff continua (metric continua) satisfy that $H(X)$ weakly contractible (pseudo-contractible) implies $H(X)$ contractible?*

As a particular question we have the following.

QUESTION 5.7. *Does pseudo-contractibility of $F_n(X)$ imply contractibility of $F_n(X)$?*

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