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MEANS ON DENDROIDS

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Abstract. Let X be a continuum. A *mean* is a continuous function $m : X \times X \to X$ such that $m(x,x) = x$ and $m(x,y) = m(y,x)$ for every $x, y \in X$. In this note we give an example to respond negatively a question that appears in [3] and observe that using a theorem that appears in [1], two other questions posed in [3] are answered.

Key words: means, dendroids, monotone maps.
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1. INTRODUCTION

A *continuum* is a non degenerated compact connected metric space. The symbol I denotes the unit interval $[0,1]$. An arc is a space homeomorphic to I. A map is a continuous function. A *mean* on a continuum X is a map $m : X \times X \to X$ such that $m(x,x) = x$ and $m(x,y) = m(y,x)$ for every $x, y \in X$. Means have been studied for several authors, basic information on this topic is found in [2], [3].

The basic problem to consider this issue is knowing which spaces admit a mean. Much progress on this issue can be found in [3]. Another line of research on this issue is to determine whether the continuous images of a continuum that admits a mean also admit a mean, i.e., we have the following problem

PROBLEM 1. *If $f : X \to Y$ is a map and X admits a mean, then $Y = f(X)$ admits a mean?*

Given a map $f : X \to Y$ between spaces X and Y, a map $h : Y \to X$ is called a *right inverse* of f provided that the composition $fh : Y \to Y$ is the identity on Y. If, for a given f, there exists a right inverse of f, then f is called an *r-mapping*.

Regarding the previous problem, in [2] the author obtained the following results:

1. If a space X admits a mean and $f : X \to Y$ is an $r-$mapping, then Y also admits a mean.

2. If a space admits a mean, then each retract of it also admits a mean.

A continuum X is said to be *hereditarily unicorehent* if each pair of subcontinua of X have intersection connected. A *dendroid* is a hereditarily unicoherent and arcwise connected continuum. An onto map between continua $f : X \to Y$ is said to be *monotone* provided that $f^{-1}(q)$ is connected for each $q \in Y$.

In [3], the following question appears

QUESTION 1 [3, Question 3.37, p. 67]. *Let a hereditarily unicoherent continuum X admit a mean, and let a mapping $f : X \to f(X)$ be monotone. Does it follow that $f(X)$ also admits a mean? If not, is the implication true under an additional assumption that X is a dendroid?*

In the Section 3 of this paper we give a negative answer to this question.

Let A be a subcontinuum of a continumm X and let $a \in A$. A pair of sequences $\{E_n : n \in \mathbb{N}\}$ and $\{F_n : n \in \mathbb{N}\}$ of subcontinua of X is called *a pair of surrounding sequences for A with respect to a* provided that:

a) $E_n \cap F_n \neq \emptyset$ for each $n \in \mathbb{N}$;

b) $A \subset \lim E_n \cup \lim F_n$;

c) $\lim(E_n \cap F_n) = \{a\}$.

With regard to this concept, in [3] is the following.

QUESTION 2 [3, Question 3.34, p. 67]. *Let a hereditarily unicoherent continuum X contains a subcontinuum A, and let two pairs of surrounding sequences $(\{E_n\},\{F_n\})$ and $(\{G_n\},\{H_n\})$ of A with respect to distinct points a and b, correspondingly, be given. Assume that the irreducible continuum between the points a and b is contained in the intersections $\lim E_n \cap \lim F_n$ and $\lim G_n \cap \lim H_n$. Does it follow that then X admits no mean?*

Additionally, in [3] the following question appears, where D is the continuum defined in Section 3.

QUESTION 3 [3, Question 3.31, p. 66]. *Does the dendroid D admits any mean?*

Making use of Theorem 3.5 of [1] we respond to these questions.

2. ANSWERS TO QUESTIONS 2 AND 3

A map f between two subspaces of X is said to be an ε - *idy map* if for all x, $d(x, f(x)) < \varepsilon$. A sequence of arcs $\overline{a_nb_n}$ is said to be *strongly converge to an arc* $\overline{ab}$ if $(\forall \varepsilon > 0)(\exists n')(\forall n \geq n')(\exists$ an ε - idy map $h : \overline{ab} \to \overline{a_nb_n})$ such that $h(a) = a_n$ and $h(b) = b_n$.

The following theorem appears in [1].

THEOREM 1 [1, Theorem 3.5]. *Let X be a compact metric space with metric d. If X contains an arc $A = \overline{ab}$ and four sequences of arcs $\overline{a_nc_n}$, $\overline{a_nd_n}$, $\overline{e_nb_n}$ and $\overline{f_nb_n}$ with each of these sequences strongly converging to A and X is such that for every n, every subcontinuum containing c_n and d_n contains a_n and every subcontinuum containing e_n and f_n contains b_n, then X does not have a mean.*

Note that this theorem gives an affirmative answer to QUESTION 2.

EXAMPLE 1. *Let (ρ, θ) denote a point of the Euclidean plane having ρ and θ as its polar coordinates. Put, for $n \in \mathbb{N}$,*

$$p = (0,0), \quad a = (1,0), \quad b = \left(1, \frac{\pi}{2}\right), \quad c = (1, \pi),$$

$$a_n = \left(1, \frac{1}{n}\right), b_n = \left(1 + \frac{1}{n}, \frac{\pi}{2}\right), p_n = \left(\frac{1}{n}, \frac{\pi}{4}\right), p'_n = \left(\frac{1}{n}, \frac{3\pi}{4}\right).$$

Define

$$D_1 = \overline{ac} \cup \overline{pb} \cup \left(\bigcup\left\{\overline{a_np_n} \cup \overline{p_nb_n} \cup \overline{b_np'_n} \cup \overline{p'_nc} \,:\, n \in \mathbb{N}\right\}\right).$$

Denote by h the reflection map about the origin, and put

$$D = D_1 \cup h(D_1), \quad \textit{(see Fig. 1).}$$

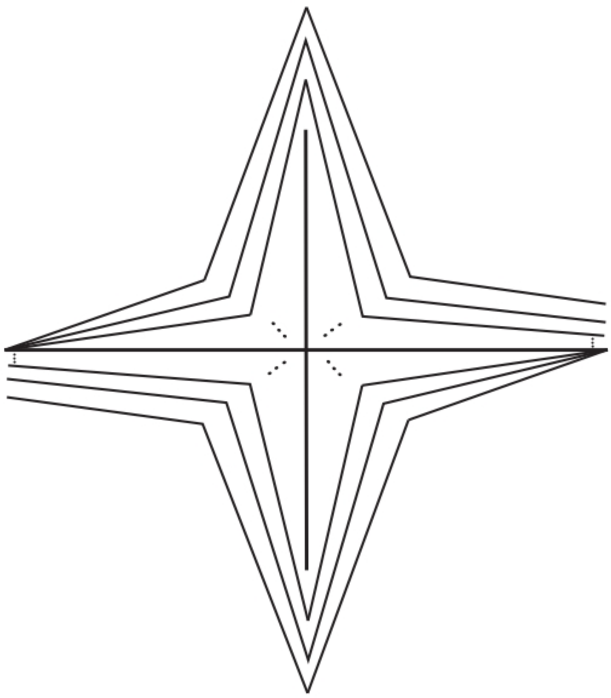

Fig. 1 – Continuum D.

Let $A = \overline{cb}$, the sequences of arcs $\overline{cb_n}$, $\overline{p_n c}$ if n is odd and the sequences of arcs $\overline{cb_n}$, $\overline{p_n c}$ if n is even, then by THEOREM 1 the dendroid D does not admit a mean; which answers QUESTION 3.

3. ANSWER TO QUESTION 1

In this section we answer QUESTION 1. To do this, in EXAMPLE 2, we construct a dendroid X that admits a mean and in EXAMPLE 3, we give a dendroid Y that does not admit a mean. We conclude by showing a monotone map from X onto Y. Finally, we ask a question related to means on X.

Let X be a compact metric space. If A is an arc in X with end points a and b then A is denoted by $\overline{ab}$.

EXAMPLE 2. *In the Euclidean plane, let* $x = (0,0)$, $y = (1,0)$ *and* $z = \left(\frac{1}{2}, 0\right)$. *For each* $n \in \mathbb{N}$, *let* $y_n = \left(1, \frac{1}{n}\right)$, $z_n = \left(\frac{1}{2}, \frac{1}{2n+1}\right)$, $w_n = \left(\frac{1}{2}, \frac{1}{2n}\right)$, $Y_n = \overline{xy_n}$, $Z_n = \overline{y_n z_n}$ *and let* $X_n = Y_n \cup Z_n$. *Let* $X' = \overline{xy}$ *and* $X = X' \cup \bigcup_{i=1}^{\infty} X_n$, (see Fig. 2).

We consider the following regions:

$$A_1 = \left\{ (x,y) \in I \times I : 0 \le x \le \frac{1}{2},\ 0 \le y \le x \right\},$$

$$A_2 = \left\{ (x,y) \in I \times I : 0 \le x \le \frac{1}{2},\ x \le y \le \frac{1}{2} \right\},$$

$$A_3 = \left\{ (x,y) \in I \times I : \frac{1}{2} \le x \le 1,\ 0 \le y \le 1 - x \right\},$$

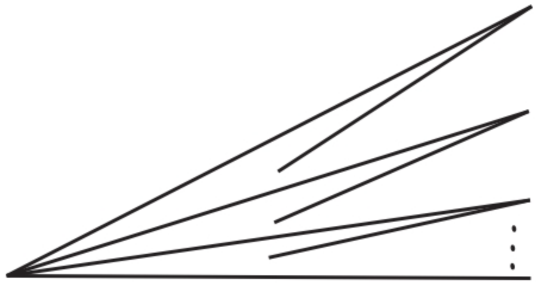

Fig. 2 – Continuum X.

$$A_4 = \left\{(x,y) \in I \times I : 0 \le x \le \frac{1}{2},\ \frac{1}{2} \le y \le 1-x\right\},$$

$$A_5 = \left\{(x,y) \in I \times I : \frac{1}{2} \le x \le 1,\ 1-x \le y \le \frac{1}{2}\right\},$$

$$A_6 = \left\{(x,y) \in I \times I : 0 \le x \le \frac{1}{2},\ 1-x \le y \le 1\right\},$$

$$A_7 = \left\{(x,y) \in I \times I : \frac{1}{2} \le x \le 1,\ \frac{1}{2} \le y \le x\right\},$$

$$A_8 = \left\{(x,y) \in I \times I : \frac{1}{2} \le x \le 1,\ x \le y \le 1\right\}.$$

We define the map $m' : I \times I \to I$ as

$$m'(x,y) = \begin{cases} y & \text{if } (x,y) \in A_1, \\ x & \text{if } (x,y) \in A_2, \\ y & \text{if } (x,y) \in A_3, \\ x & \text{if } (x,y) \in A_4, \\ x+2y-1 & \text{if } (x,y) \in A_5, \\ 2x+y-1 & \text{if } (x,y) \in A_6, \\ x & \text{if } (x,y) \in A_7, \\ y & \text{if } (x,y) \in A_8. \end{cases}$$

It is easy to see that m' is a mean on I. For simplicity, we will identify the function $m' : I \times I \to I$ by $m : X' \times X' \to X'$. Let $r : X \to X'$ be the retraction $r(x,y) = (x,0)$. For each $n \in \mathbb{N}$, we consider the maps $f_n = (r|_{Y_n})^{-1} : \overline{xy} \to Y_n$ and $g_n = (r|_{Z_n})^{-1} : \overline{zy} \to Z_n$. Note that f_n and g_n are homeomorphisms.

We define the map $m_n : X_n \times X_n \to X_n$ as

$$m_n(x,y) = \begin{cases} f_n(m(r(x), r(y)) & \text{if } (x,y) \in (Y_n \times Y_n) \cup (Y_n \times Z_n) \cup (Z_n \times Y_n), \\ g_n(m(r(x), r(y)) & \text{if } (x,y) \in Z_n \times Z_n. \end{cases}$$

It is easy to see that m_n is a mean on X_n.

Now we define the map $M : X \times X \to X$ as

$$M(x,y) = \begin{cases} m_n(x,y) & \text{if } (x,y) \in (X_n \times X_n) \cup (X' \times X_n) \cup (X_n \times X'), \\ m(x,y) & \text{if } (x,y) \in (X' \times X') \cup (X_n \times X_m),\ n \ne m. \end{cases}$$

Since m_n and m are means, M is a mean on X.

EXAMPLE 3. *In the Euclidean plane let* $s = (0,0), t = (1,0), U = \overline{st}$. *For each* $n \in \mathbb{N}$, *let* $t_n = \left(1, \frac{1}{n}\right), t'_n =$

$\left(\frac{n+1}{n},0\right), s_n = \left(0,-\frac{1}{n}\right)$, $T_n = \overline{st_n}, T_n' = \overline{t_n t_n'}$, $S_n = \overline{t_n' s_n}$ *and let* $U_n = T_n \cup T_n' \cup S_n$. *Let* $Y = U \cup \bigcup_{i=1}^{\infty} U_n$, (see Fig. 3).

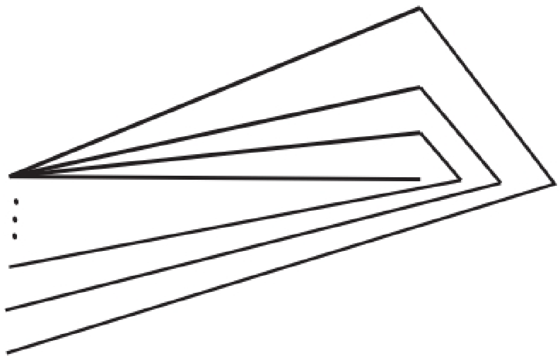

Fig. 3 – Continuum Y.

We consider $U = \overline{st}$, the sequences of arcs $\overline{st_n'}, \overline{t_n' s_n}$, $\overline{ss_{2n}}$ and $\overline{ss_{2n-1}}$ for every n. Then by THEOREM 1, the continuum Y does not admit a mean.

To conclude the answer to Question 1, we consider the subcontinuum W of X, $W = \overline{xz} \cup \bigcup_{i=1}^{\infty} \overline{xw_i}$, we define the map $h : X \to Y$ identify W in the point $(0,0)$. Note that this map is monotone, also X admits and Y does not admit a mean.

To finish this paper we have the following.

An onto map between continua $f : X \to Y$ is said to be *confluent* provided that for each subcontinuum B of Y and each component C of $f^{-1}(B)$, $f(C) = B$.

Now, we consider the continuum X of EXAMPLE 2, we will see that the mean M is not confluent.

Let $p,q \in X'$ with $p = (p_1,0)$ and $q = (q_1,0)$ where $0 < p_1 < \frac{1}{2} < q_1 < 1$. Notice that $C = \overline{z_1 g_1(q)} \times \overline{z_2 g_2(q)}$ is a component of $M^{-1}(\overline{pq})$, in fact

$$\begin{aligned} M^{-1}(\overline{pq}) \quad = \quad & \left(\bigcup_{\substack{m,n=1 \\ m\neq n}}^{\infty} \overline{f_n(p) f_n(q)} \times \overline{f_m(p) f_m(q)} \right) \cup \\ & \left(\bigcup_{\substack{m,n=1 \\ m\neq n}}^{\infty} \overline{z_n g_n(q)} \times \overline{z_m g_m(q)} \right) \cup \\ & \left(\bigcup_{\substack{m,n=1 \\ m\neq n}}^{\infty} \overline{f_n(p) f_n(q)} \times \overline{z_m g_m(q)} \right) \cup \\ & \left(\bigcup_{\substack{m,n=1 \\ m\neq n}}^{\infty} \overline{z_m g_m(q)} \times \overline{f_n(p) f_n(q)} \right) \end{aligned}$$

But $M(C) \neq \overline{pq}$.

By the above, we have the following question

QUESTION 4. *Does the continuum X, of EXAMPLE 2, admit confluent means?*

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